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Reconstruction of the gluon density in the proton as an inverse problem

Inverse Days 2025

Henri Hänninen

December 17, 2025

Based on joint work with

Antti Kykkänen (Rice U.) & Hjørdis Schlüter (Helsinki U.)

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Applying inverse problems and abstraction to particle physics

PHYSICAL REVIEW D **112,** 094026 (2025)

Reconstruction of the dipole amplitude in the dipole picture as a mathematical inverse problem

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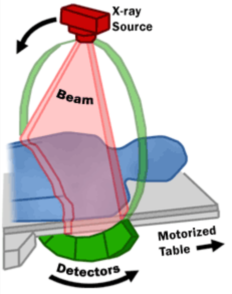
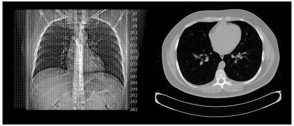
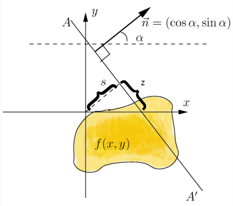
We show that the inference problem of constraining the dipole amplitude with inclusive deep inelastic scattering data can be written into a discrete linear inverse problem, in an analogous manner as can be done for computed tomography. To this formulation of the problem, we apply standard inverse problems methods and algorithms to reconstruct known dipole amplitudes from simulated reduced cross section data with realistic precision. The main difference of this approach to previous works is that this implementation does not require any fit parametrization of the dipole amplitude. The freedom from parametrization also enables us for the first time to quantify the uncertainties of the inferred dipole amplitude in a novel more general framework. This mathematical approach to small-x phenomenology opens a path to parametrization bias-free inference of the dipole amplitude from HERA and Electron-Ion Collider data.

DOI: 10.1103/7fhf-7fp4

Deep inelastic scattering analogous with computed tomography?







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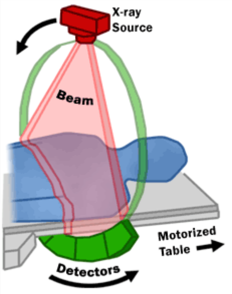
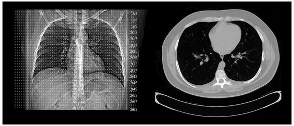
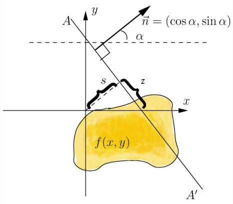
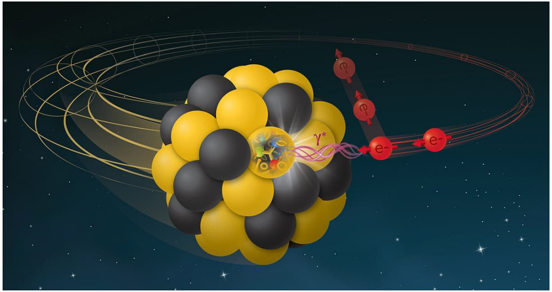
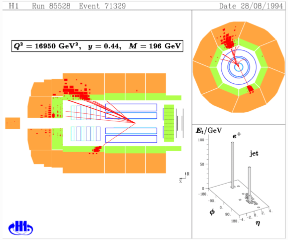


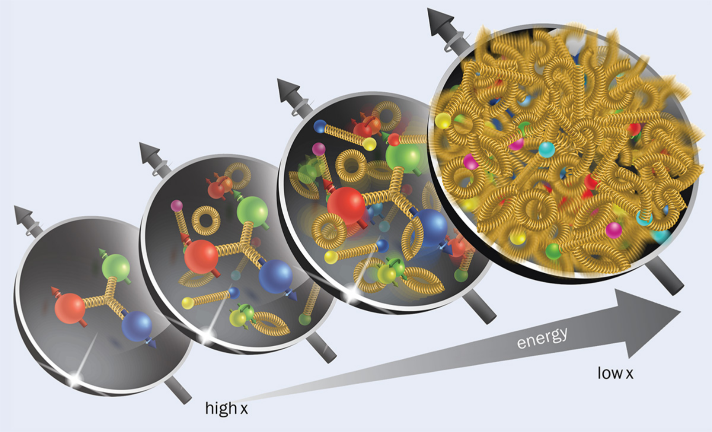








Proton is composed of quarks which radiate gluons which radiate gluons which radiate gluons which radiate gluons which radiate...



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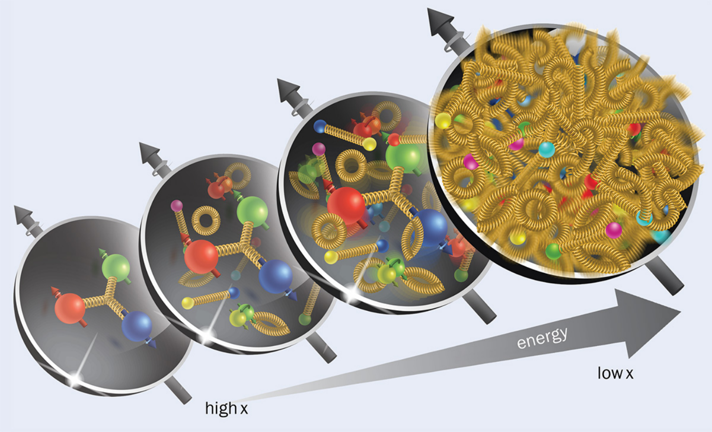

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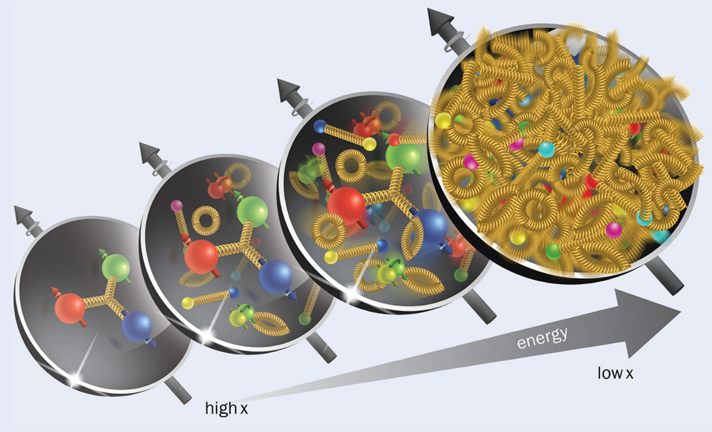

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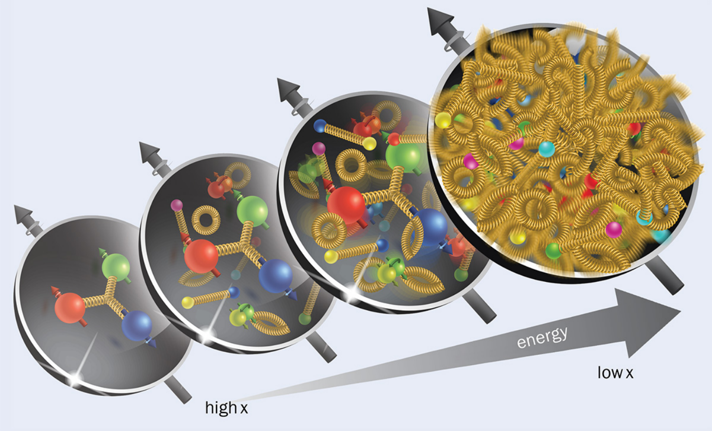

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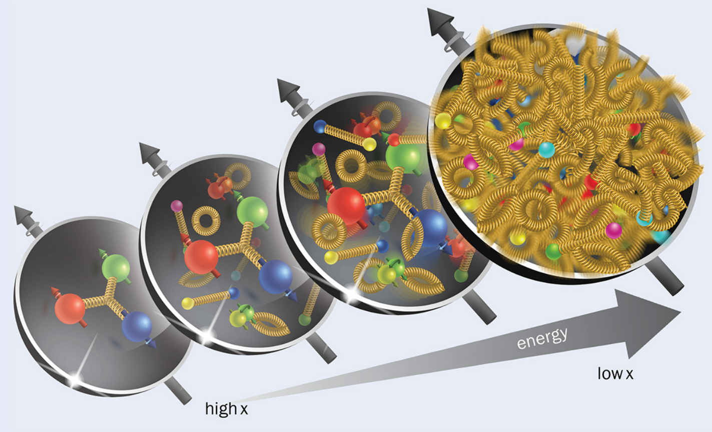

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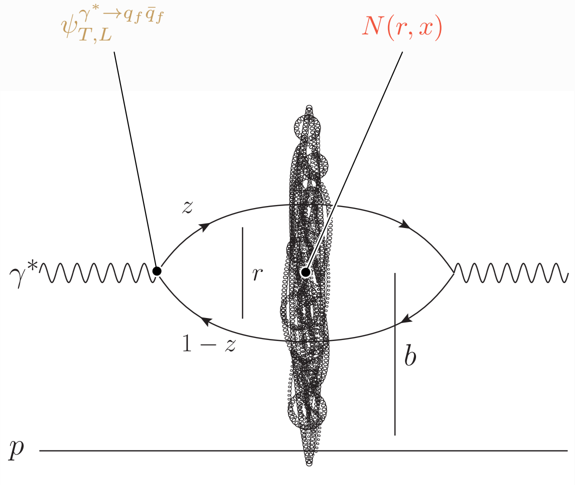
New particle collider in construction at BNL: Electron–Ion Collider

A key goal of the Electron–Ion Collider is to uncover this process precisely (2030s–).

Forward problem: Dipole picture of electron–proton DIS



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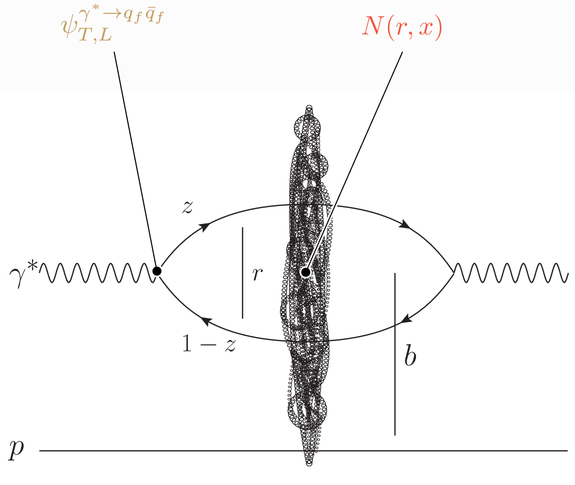

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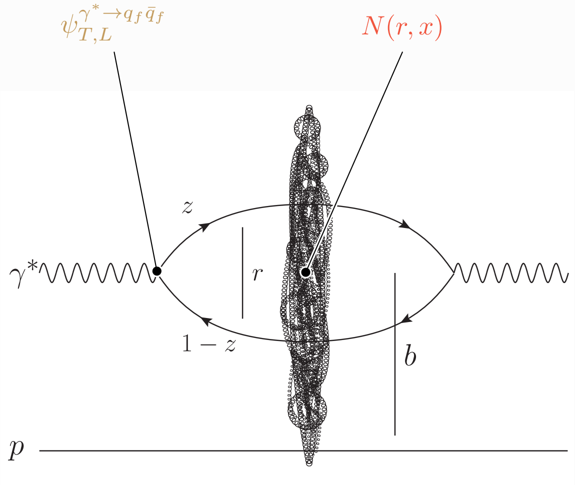

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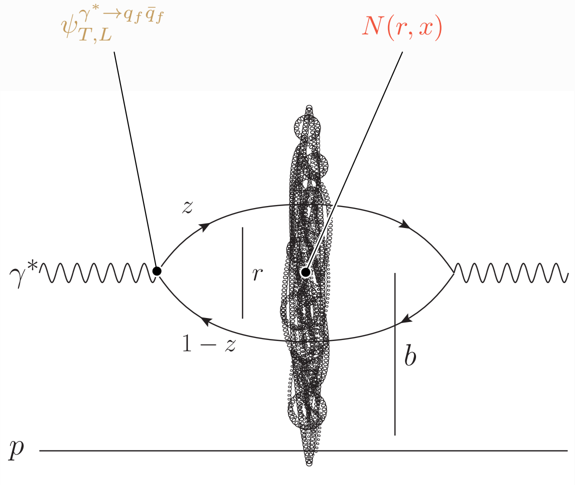

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- Quark-antiquark dipole state scatters off the gluon cloud via strong nuclear interaction, quantified by the dipole amplitude $N(r, x)$.
- $\psi_{T,L}^{\gamma^* \to q_f \bar{q}_f}$ from QED and well understood, $N(r, x)$ is related to dense and many-body quantum structure of the proton in non-perturbatively strong interaction (hard to understand precisely in QCD theory).

Reduced cross section data to constrain the dipole amplitude

Reduced cross section σ_r is defined in the dipole picture at leading order accuracy as:

$$\sigma_r(y, x, Q^2) = F_T(x, Q^2) + \frac{2(1-y)}{1+(1-y)^2} F_L(x, Q^2)$$

$$F_{L,T}(x, Q^2) = \frac{Q^2}{4\pi\alpha_{\mathrm{em}}} \sigma_{L,T}(x, Q^2)$$

$$y = \frac{Q^2}{xs}$$

$$\sigma_{T,L}(x, Q^2) = \frac{\sigma_0}{2} \frac{1}{4\pi} \sum_f \int_{\mathbb{R}^2} \int_0^1 \left| \psi_{T,L}^{\gamma^* \to q_f \bar{q}_f}(\mathbf{r}, Q^2, z, f) \right|^2 N(\mathbf{r}, x) \mathrm{d}^2\mathbf{r} \mathrm{d}z,$$

which is an implicit inverse problem for $N(r, x)$, conventionally solved by fitting a theoretically motivated model parametrization for N, such as the McLerran–Venugopalan model.

Rewriting into an explicit inference problem

We rewrite the reduced cross section in the dipole picture as:

$$\sigma_r(x, Q^2) = \int_0^{\infty} Z(r, Q^2, y(Q^2)) \frac{\sigma_0}{2} N(r, x) \mathrm{d}r,$$

with the definitions:

$$Z(r, Q^2, y) := \frac{rQ^2}{4\pi\alpha_{\mathrm{em}}} \left[Z_T(r, Q^2) + \frac{2(1-y)}{1+(1-y)^2} Z_L(r, Q^2) \right],$$

$$Z_{T,L}(r, Q^2) = \sum_f \int_0^1 \left| \psi_{T,L}^{\gamma^* \to q_f \bar{q}_f}(r, Q^2, z, f) \right|^2 \mathrm{d}z.$$

- The key assumption was to require $y = y(Q^2, x, s) \equiv y(Q^2)$, i.e. that s and x are constant.
- Looks like an **integral transform** of the dipole amplitude, which we can solve as a linear inverse problem. This hasn't been realized as a practical possibility before.

Discrete linear inverse problem

We discretize the r-integral and the problem becomes discrete and linear inverse problem:

$$\sigma^{\mathrm{d}}_{r,j} := \sigma^{\mathrm{d}}_r(Q^2_j, x) = \varsigma^i_j \mathbf{n}_i$$

is an explicit reconstruction problem of the form $\mathbf{b} = \mathbf{A}\mathbf{x}$, to which we may apply standard methods, such as Tikhonov regularization.

- *This* is the aspect we have in common with CT, which can / has been written in this form, and solved with these methods.
- The forward operator ς is now completely fixed by the (light-cone) perturbation theory and the experiment via the definition of the reduced cross section, and all non-perturbative degrees of freedom are in $\mathbf{n}_i := \frac{\sigma_0}{2} N_i$, to be solved by reconstruction.

Numerical reconstruction and regularization

We use 1st order derivative operator Tikhonov regularization, where we minimize the cost function

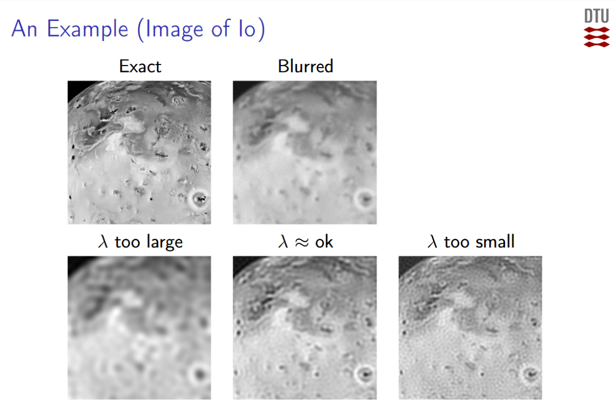
$$\arg\min_{\mathbf{n}\in\mathbb{R}^M} \{||\varsigma\mathbf{n} - \sigma_r||_2^2 + \lambda||\frac{\mathrm{d}}{\mathrm{d}r}\mathbf{n}||_2^2\},$$

where the first is the "data fitting term" and the second is the regularization term. The regularization parameter λ tunes the size of the penalization of a large value for the first derivative of $\mathbf{n}$.

- Compared to 0th, 2nd ord. Tikhonov, and 0th–2nd order preconditioned Kaczmarz and Cimmino methods.

Implementations of regularization algorithms by AIR Tools II and regtools:
https://github.com/jakobsj/AIRToolsII,
https://www2.compute.dtu.dk/~pcha/Regutools/

Regularization parameter λ in $\mathcal{O}(30)$ words

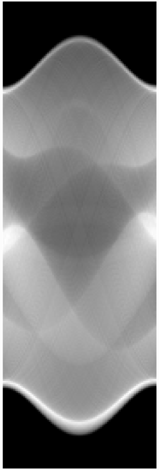


More on the choice of regularization parameter at `http://www2.imm.dtu.dk/~pcha/DIP/chap5.pdf`.

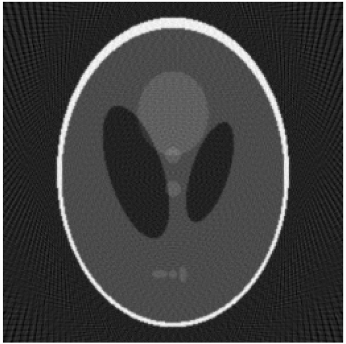
- λ important: affects reconstruction quality.
- Choosing λ difficult:
  - Big λ bad: smudge, small λ bad: noisy.
  - Middle λ good, but *where middle?*
- No "ground truth" for dipole, need unsupervised λ:
  - We use $\mathrm{err}_{\mathrm{noiseless}}$ and err_{χ^2}.
  - Need more robust methods: discrepancy principle, L-curve criterion, GCV criterion, NCP criterion.

Closure test: reconstruction of the Shepp–Logan phantom

Shepp–Logan phatom: a standard test image.



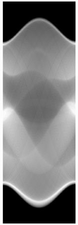
Radon transform of the phantom.



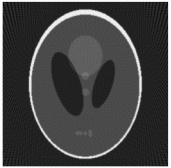
Inverse Radon transform (reconstruction) from the sinogram.

Choosing a ”phantom“ of the dipole amplitude

Forward pr. →



Reconstr. →



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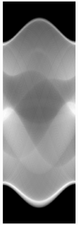

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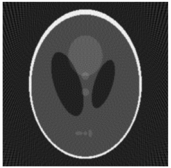

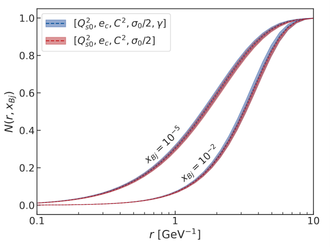

Casuga–Karhunen–Mäntysaari reference dipole

Choosing a ”phantom“ of the dipole amplitude

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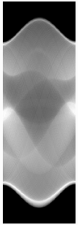

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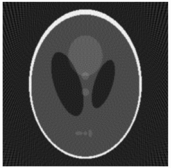

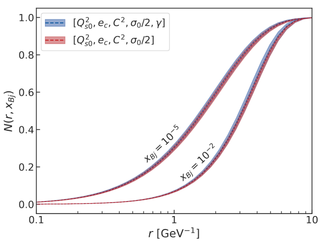

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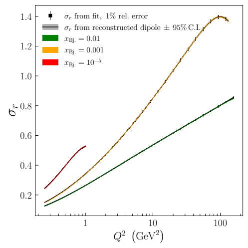
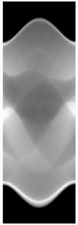


DIS reduced cross section from CKM dipole

Choosing a ”phantom“ of the dipole amplitude

Forward pr.



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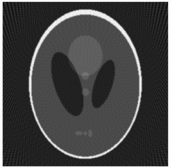

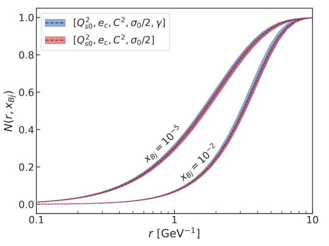

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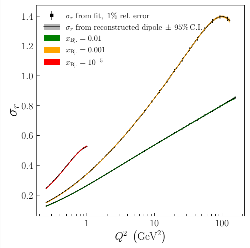

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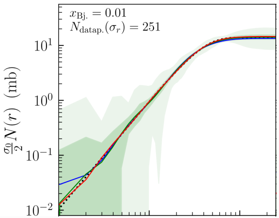
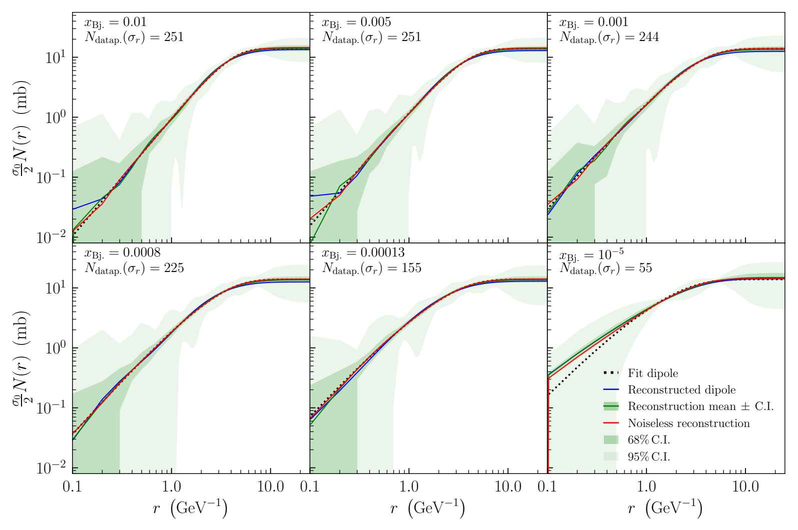


Reconstruction

Reconstruction results: light quarks only, 4-par-CKM-dipole



- Overall the reconstruction is working quite well.
- Growing uncertainty at small-r reflects that the reduced cross section data is not sensitive to the dipole amplitude in that regime, which is expected.
- At small x with the fewest datapoints, the reconstruction loses accuracy.
  - $Q^2 = s x_{\mathrm{Bj}} y \leq s x_{\mathrm{Bj}}$

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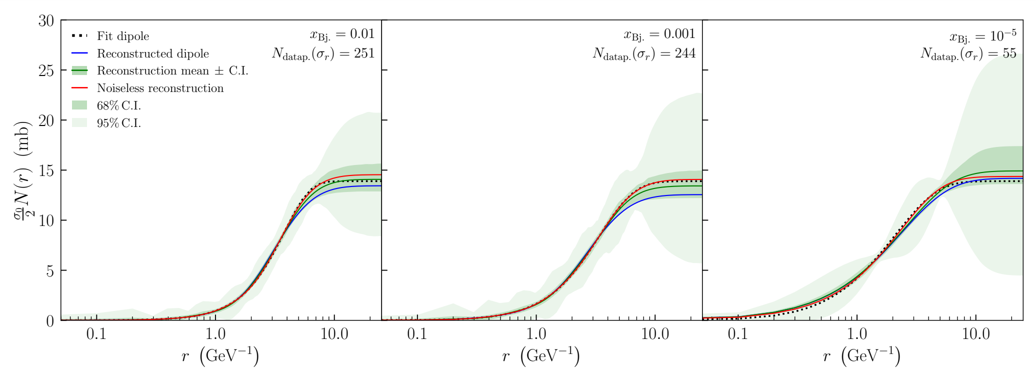
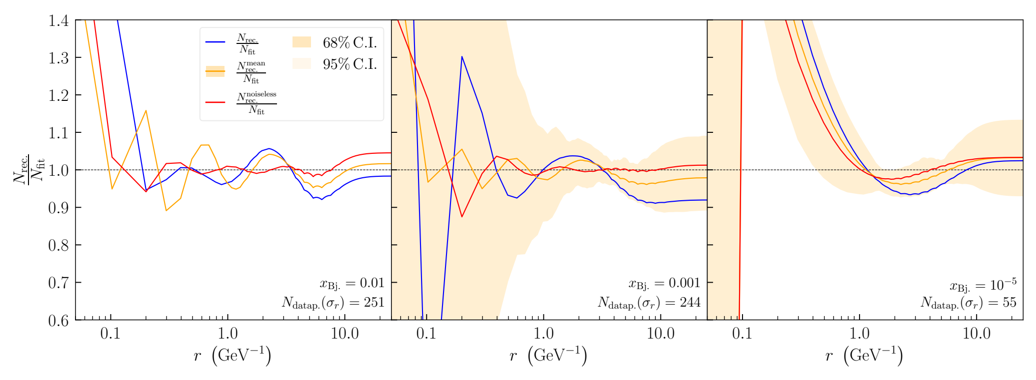


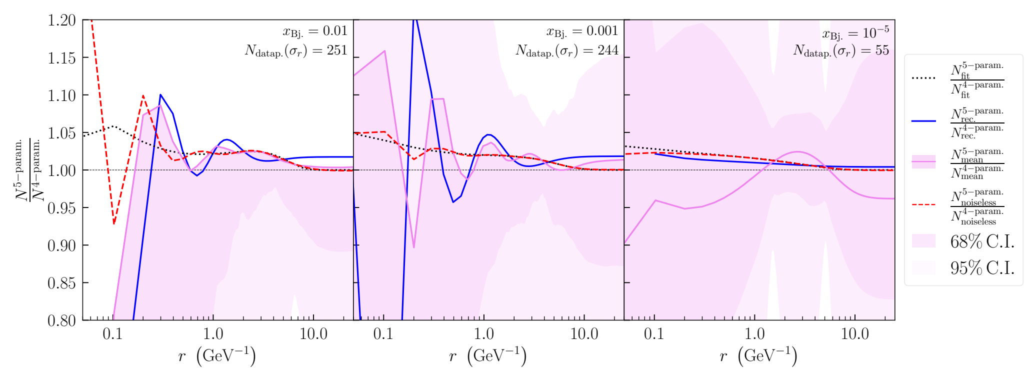
- Linear y-axis: at large-r we see that accuracy is reduced due to the exponentially vanishing forward operator ς. With sufficient data it is working to some degree. Would be very exciting to get accurate reconstruction at large r: peak size is related to proton transverse area.

Reconstruction of the 5-parameter reference dipole



- Performance of the reconstruction is very similar for the second reference dipole as well. The algorithm is completely agnostic of the original parametrization of the reference dipole.

Can it differentiate between the two parametrizations?



- Proof-of-principle: in ”perfect“ (noiseless) conditions, the ratio is recovered quite faithfully.
- Given sufficient data the reconstruction is able to reproduce the ratio of the two reference fit dipole amplitudes. With reduced number of datapoints and the addition of noise (error) in the data, the reconstruction begins to deteriorate or fail.

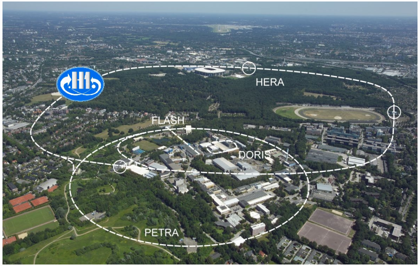
Reconstruction from real data

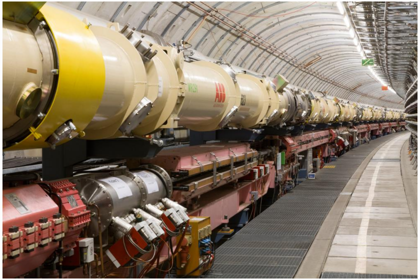
What about real data!?

- The dipole amplitude has never been measured, only constrained by theoretically motivated fits. Literature might even suggest that it *cannot* be measured.
- Question: When is a reconstruction as a solution to an inverse problem good enough to constitute a measurement of the inferred quantity?
  - The framework of inverse problems might be powerful enough to push the frontier of what is considered as measurable in elementary particle physics!

HERA data

- Electron–proton DIS data is available from HERA (*Hadron-Elektron-Ringanlage, Hadron–Electron Ring Accelerator*)
  - Operated in Hamburg, Germany, from 1992 to 2007.
  - Electrons (or positrons) were collided with protons at a center-of-mass energy of $320\,\mathrm{GeV}$.
  - Relativistic collision: both particles are moving very nearly at the speed of light.
  - If the proton was at rest instead, the same CoM energy would require the electron to have $59\,\mathrm{TeV}$ of kinetic energy. (~ 63000 times the rest energy contained by the mass of the proton, $E = mc^2$.)





Available HERA data

We need $\sigma_r(Q^2, x, y = y(s, x, Q^2))$ at **fixed** s and x: **much less data** than in closure test.

HERA data[a] is available in four bins of $\sqrt{s}$ (center of mass frame energy):

$\sqrt{s}$	N
318.1	**644**
300.3	112
251.5	260
224.9	210

Data at lower $\sqrt{s}$ almost exclusively at large x, and bins are small.

[a] H1 and ZEUS Collab., H. Abramovicz et al., Eur.Phys.J.C75 (2015) 12, 580

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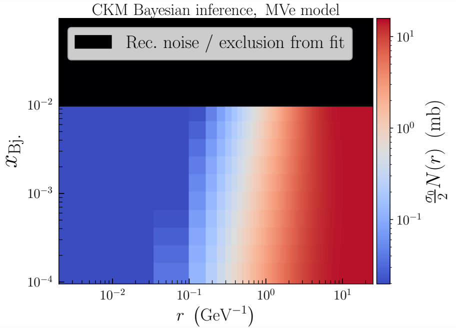
At $\sqrt{s} = 318\,\mathrm{GeV}$, $x_{\mathrm{Bj.}} <= 0.08$, all bins with $N \geq 6$ in Q^2:

$x_{\mathrm{Bj.}}$	$N(\sigma_r(\sqrt{s}, x_{\mathrm{Bj.}}))$
0.00013	*8*
0.0002	9
0.00032	11
0.0005	15
0.0008	19
0.0013	18
0.002	21
0.0032	24
0.005	24
0.008	23
0.013	30
0.02	*35* (vs. $N \in [55, 251]$)
0.032	33
0.05	30
0.08	30

HERA data preliminary results disclaimer

- Basically, we're testing how well the existing implementation works on real data to see what works, and what doesn't.
- Multiple planned improvements, beyond the scope of this talk, unfortunately.

Visualizing the dipole amplitude data: Reference dipole



- The CKM reference dipole shown in a $(r, x_{\mathrm{Bj.}})$ plane.
- Initial condition of the CKM dipole amplitude parametrized according to the MVe model:

$$N(r, x = 0.01) = 1 - e^{-\frac{(r^2 Q_{s,0}^2)^\gamma}{4} \ln\left(\frac{1}{r\Lambda_{\mathrm{QCD}}} + e_c e\right)}.$$

- An ODE (BK eq.) describes the evolution in $x_{\mathrm{Bj.}}$.

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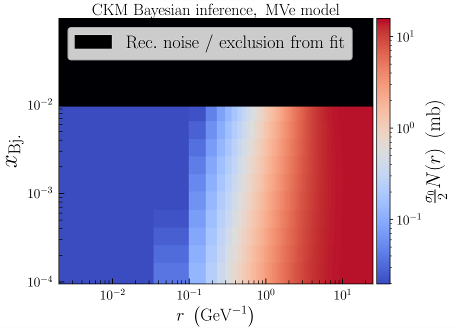

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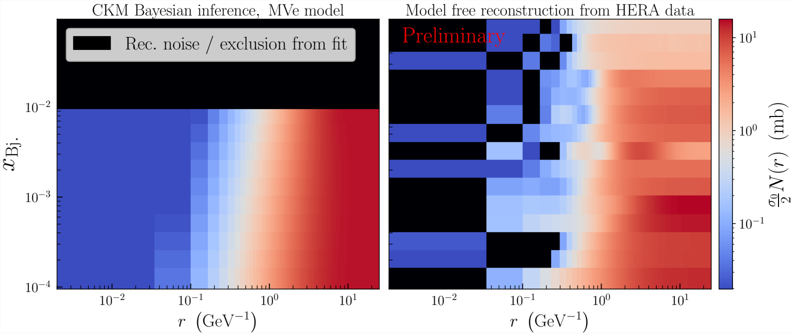
- An ODE (BK eq.) describes the evolution in $x_{\text{Bj.}}$.
- Uniform target size $\frac{\sigma_0}{2}$ has been assumed to be reasonable for inclusive DIS cross section data.
  - Transverse momentum exchange in the collision is not measured, so it is thought that inclusive DIS is not sensitive to the target size.

Preliminary HERA data reconstruction, only $\sqrt{s} = 318\,\mathrm{GeV}$



- The reconstruction recovers a noisy resemblance of the CKM fit dipole.
  - Units, length: $1\,\mathrm{GeV}^{-1} = 0.197\,\mathrm{fm} \approx 2 \cdot 10^{-15}\,\mathrm{m}$. Area: $1\,\mathrm{mb} = 10^{-3}\,\mathrm{barn} = 10^{-31}\,\mathrm{m}^2$.
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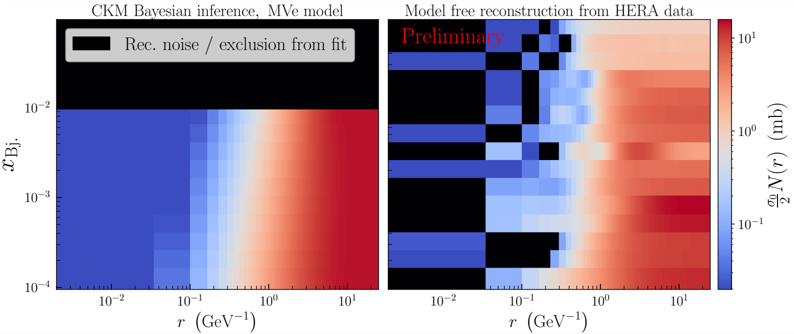

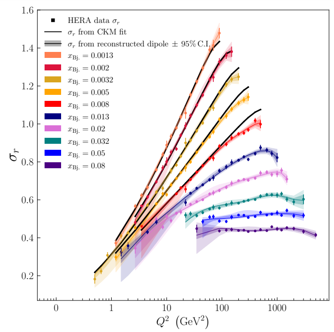

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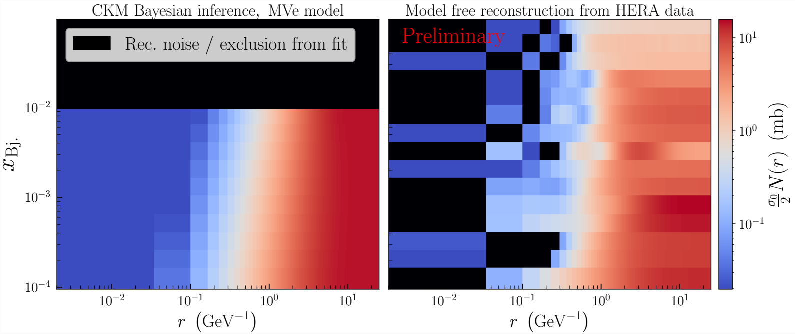

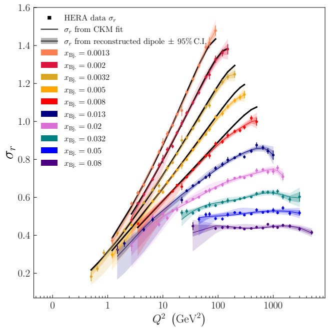

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  - Reconstruction works outside of conventionally considered regime at $x_{\mathrm{Bj.}} > 0.01$. How to interpret the reconstructed dipole amplitude in terms of QCD theory?
  - Reconstruction enables the computation of quantities from HERA data without an assumed model(!) (saturation scale, unintegrated gluon distribution, proton size $\sigma_0(x_{\mathrm{Bj.}})/2$?)

Conclusions

- We wrote the dipole picture of DIS into an integral transform of the dipole amplitude, and showed that it can be inverted with standard numerical inverse problems methods.
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Conclusions

- We wrote the dipole picture of DIS into an integral transform of the dipole amplitude, and showed that it can be inverted with standard numerical inverse problems methods.
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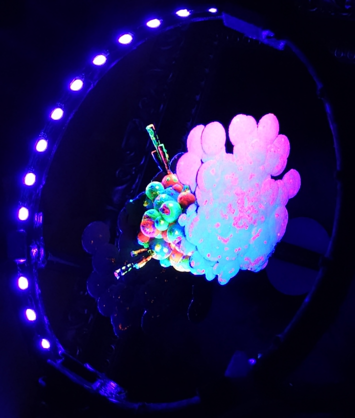
- Closure test demonstrates applicability to real data, at least with the Electron–Ion Collider, and with more advanced reconstruction methods perhaps already with HERA data.
  - Preliminary attempt at reconstruction from HERA data seems promising, and we already know how to do better. (Log grid, non-neg., smoothness in $x_{\mathrm{Bj.}}$, more data via multimodality, ...)
- Reconstruction of the dipole amplitude *immediately* enables the extraction of physical information about the gluon saturation process, that has not been seen in a model free way.
- Inverse problems mathematics and methods can open new paths towards novel inference in particle and high-energy physics.
  - Perhaps even to measure quantities never measured before?
  - When is a reconstruction ”good enough“ to constitute a measurement?



Thank you for your attention!

Questions?

Contact:

henri.j.hanninen@jyu.fi

Slides:

https://hhannine.github.io/talks

An all encompassing review of Inverse Problems?

- When we began our work on the dipole amplitude problem we didn't know what kind of a problem it would be, or what would be useful methods to solve it.
  - The comparison to CT has developed after the original work.
- In particle physics there is the *Review of Particle Physics*
  - S. Navas *et al.* (Particle Data Group), Phys. Rev. D **110**, 030001 (2024)
  - Contributions from > 100 experts, updated periodically. 2024 edition is 2382 pages.
- Application of the inverse problems framework and methods in new problems in high-energy physics begins with a big hurdle:
  - Inference problems are abstract and complex, with no physical analogy in classical problems.
  - One needs to identify the type of an inverse problem, and to know what methods that enables one to use, and why that can be powerful. There even might be new types of IP.
    - Sound waves in elastic media vs. gravitational waves in spacetime.
  - Rough idea of the encyclopedia entry for an inverse problem type:
    - Mathematical distillation of the inverse problem. (e.g. Linear IP)
    - What are the tools to work on with such a problem. (Discretization, uniqueness, stability, fancy state of the art methods,...?)
    - Examples of applications. (CT, dipole amplitude inference from DIS, ...)
  - Gotta cover them all: Higher dimensions, tensors, abstract manifolds and transforms, PDEs, etc. (advanced applications in particle physics, cosmology, quantum mechanics,...)

Appendix: Additional slides.

Mathematical observations

We want to achieve **injectivity in a functional sense** for the mapping $N(r,x) \mapsto \sigma_r(Q,x)$:

$$\sigma_r(y, x, Q^2) = F_T(x, Q^2) + \frac{2(1-y)}{1+(1-y)^2} F_L(x, Q^2)$$

$$F_{L,T}(x, Q^2) = \frac{Q^2}{4\pi\alpha_{\mathrm{em}}} \sigma_{L,T}(x, Q^2)$$

$$y = \frac{Q^2}{xs}$$

$$\sigma_{T,L}(x, Q) = \frac{\sigma_0}{2} \frac{1}{4\pi} \sum_f \int_{\mathbb{R}^2} \int_0^1 \left| \psi_{T,L}^{\gamma^* \to q_f \bar{q}_f}(\mathbf{r}, Q^2, z, f) \right|^2 N(\mathbf{r}, x) \mathrm{d}^2\mathbf{r} \mathrm{d}z.$$

First simplifying assumptions:

- Keep x fixed (constant) in calculations. Reconstruct $N(x,r)$ at fixed x.
- Take s as constant. With these constraints $y \equiv y(Q^2)$.

Discretizing the problem for numerical inversion

We want to numerically compute the inverse of the ”dipole cross section transform“ without having an explicit formula for the inversion. Inverse problems methods enable us to do this. Taking r on a grid $\{r_0, r_1, \ldots, r_M\}$ and writing the r-integral as a Riemann sum:

$$\sigma_r^{\mathrm{d}}(x, Q^2) = \sum_{i=0}^{M} (r_{i+1} - r_i) Z(r_i, Q^2, y(Q^2)) \frac{\sigma_0}{2} N(r_i, x),$$

we write the problem as a proper linear equation:

$$\sigma_{r,j}^{\mathrm{d}} := \sigma_r^{\mathrm{d}}(Q_j^2, x) = \varsigma_j^i \mathbf{n}_i,$$

where we absorbed the interval length $(r_{i+1} - r_i)$ into the definition of ς_j^i (the forward operator[1]), and $\frac{\sigma_0}{2}$ into $\mathbf{n}_i$ (discretized dipole amplitude). This is a discrete linear inverse problem for $\mathbf{n}_i$.

[1]The symbol ς, `\varsigma`, is the *final form* lower case sigma used at the end of a word: ”f-sigma“.

Solving the inverse problem numerically

Generally the discrete problem $\sigma = \varsigma \mathbf{n}$ has

- $\mathbf{n} \in \mathbb{R}^M$: M is constrained by numerical accuracy of the computation (r grid size)
- $\sigma \in \mathbb{R}^m$: m is constrained by the available data
- $\varsigma \in \mathbb{R}^{m \times M}$: matrix valued forward operator.

If $m \equiv M$ we would have a full-rank problem (and ς could have a true inverse), but in practice we have a underdetermined problem with $m < M$.

- (i.e. there is fewer data points of σ_r available than we need to have discretization points for $N(r, x)$ to have accurate calculation of the cross sections.)

Regularization methods allow us to solve underdetermined problems, and to work with data that has noise / uncertainty.

- Computing the inverse integral transform from perfect data is easier than from data that has noise. Applications like medical imaging routinely work with noisy real world data.

Related integral transforms

The K-transform defined by

$$g(y,\nu) = \mathcal{K}_\nu[f(x);y] = \int_0^\infty (xy)^{\frac{1}{2}} K_\nu(xy) f(x) \mathrm{d}x$$

is analytically invertible. Identifying the measurement with $g(y) \sim \sigma_r(Q)$ and the dipole amplitude with $f(x) \sim N(r)$ bears a rough resemblance to the reduced cross section integral transform of the dipole amplitude.

Guess: the fact that our approach works could be related to the existence of a proper integral transform? Closer mathematical inspection needed, work in progress.

Generating data for the closure test

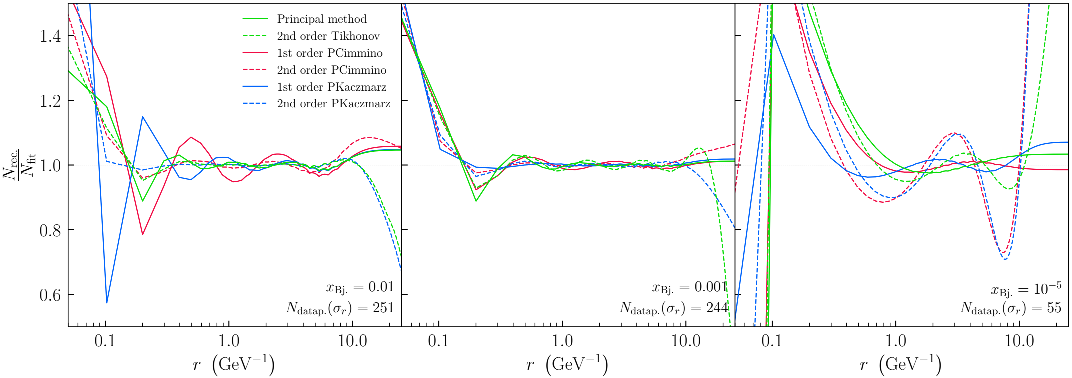
We use the four and five parameter Bayesian LO CKM inferences as the references dipoles to generate data:

- Casuga, Karhunen, Mäntysaari, Phys. Rev. **D** 109, 054018
- We generated two cross section datasets, and two corresponding forward operators:
  - Light quarks only, as in the reference CKM work.
  - Light + charm to demonstrate that the reconstruction works also with the inclusion of charm in the forward operator. Used cross section data is artificial in the sense that original fits did not include charm.

We compute in HERA kinematics with $\sqrt{s} = 318\,\mathrm{GeV}$ reduced cross section data in similar bins of Bjorken-x, but with more ample number of points in Q^2.

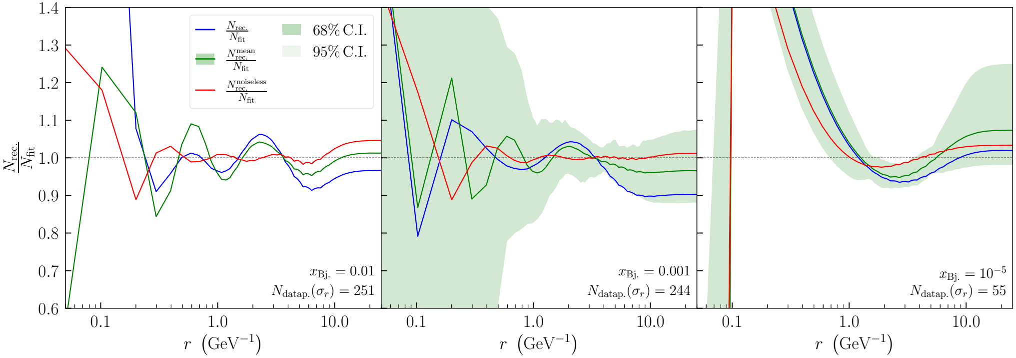
- A full rank problem with $m = M := 256$ would be the reference for an "easy" problem.
- The less data of the measurement $\sigma_r(Q^2)$ we have available, i.e. $m < M$, the more underdetermined, and harder, the problem becomes.

Comparison of regularization methods



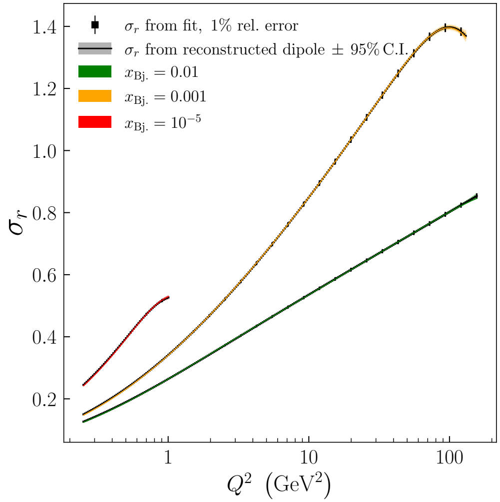
- Comparison of six best reconstruction algorithms as shown by the ratio $\frac{N_{\rm rec}}{N_{\rm fit}}$.
- Chosen ”Principal“ method, 1st order preconditioned Tikhonov, performed the best on average.

Relative accuracy of the reconstruction



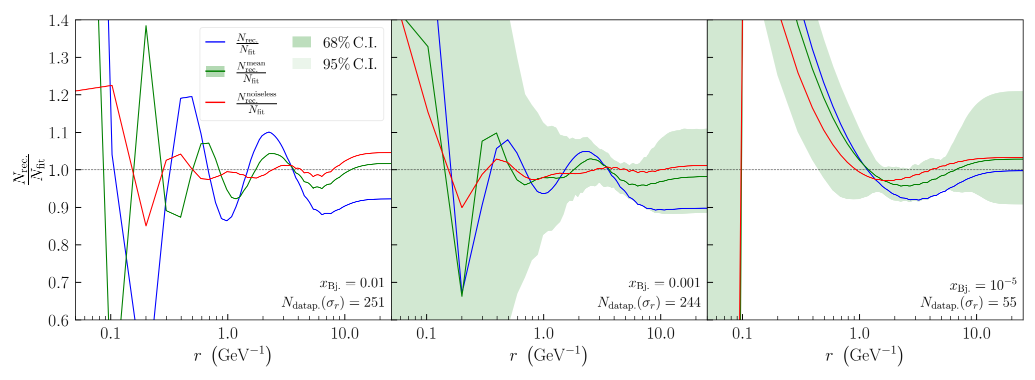
- Proof-of-principle: at least with enough data (of sufficient quantity and fidelity) this reconstructive approach is capable of reaching high accuracy, especially in the intermediate-r regime.

Cross sections from reference dipole and reconstruction



- **All** of the reconstructions from the 95% confidence intervals map into the colored 95% C.I. bands shown here [sic].
- Shows how the reduced cross section data does not constrain the dipole amplitude at small r, and only to a limited degree at large r.

Accuracy with light + charm quarks



- Addition of the charm contribution to the forward operator is a minor change that does not affect the performance of the reconstruction. The underlying problem stays linear.

HERA data preliminary results disclaimer

- Testing the implementation of the method paper without new improvements like enforcing non-negativity.
- **Very** limited quantity of HERA data in the form we need: only using the largest $\sqrt{s}$ set.
  - Some of the data is at high $Q^2 \geq 400\,\mathrm{GeV}^2$, but we use them to have any hope of having enough data points.
- Not including charm production data (yet!).
- Not leveraging the BK equation which could help with the limited data situation.
- Forward operator is only LO: we don't know how this affects the reconstruction ("model uncertainty" in inverse problems): heuristically reconstructed $N_{\mathrm{rec}} \sim \varsigma_{\mathrm{LO}}^{-1}(\sigma_r^{\mathrm{HERA}})$, so the reconstruction compensates for everything *not exactly* the LO dipole picture: beyond LO in α_s, next-to-eikonal effects, large Q^2 effects(?), and probably much more.
- Rudimentary implementation of the unsupervised regularization, we should try to get the better methods working. We might not be finding the optimal λ, which could cause surplus noise in the reconstruction, or over-smoothing.

Towards global analysis as a multimodal inverse problem

How do we use all possible data simultaneously? And what about the charm production data?
We can stack the reconstruction problems and the algorithm doesn't care:

$$\begin{bmatrix} \sigma_r^{\text{incl.}}(x, Q^2, s_1) \\ \sigma_r^{\text{incl.}}(x, Q^2, s_2) \\ \sigma_r^{\text{incl.}}(x, Q^2, s_3) \\ \sigma_r^{c\bar{c}}(x, Q^2, s_{c\bar{c}}) \end{bmatrix} = \begin{bmatrix} \varsigma_r^{\text{incl.}}(Q^2, s_1) \\ \varsigma_r^{\text{incl.}}(Q^2, s_2) \\ \varsigma_r^{\text{incl.}}(Q^2, s_3) \\ \varsigma_r^{c\bar{c}}(Q^2, s_{c\bar{c}}) \end{bmatrix} \mathbf{n}(r, x),$$

i.e. we can compute a new forward operator for each $s \in \mathbb{R}$, and include the corresponding data in the reconstruction. This also enables the inclusion of the charm production data: the forward operator is computed only with $f \equiv c$. *Work in progress.*