JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

Constructing a new mathematical inverse problem to solve a 50-year-old problem in high-energy physics

Jyväskylä Inverse Problems seminar

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September 23, 2026

Preprint arXiv:2609.11896 [hep-ph]

Objective: identify mathematical structure

(a) CT scanner

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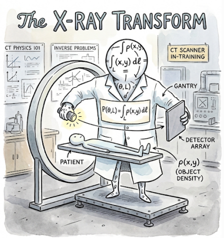

(b) CT scanner $\equiv$ X-ray transform

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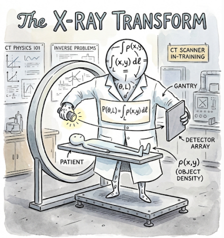

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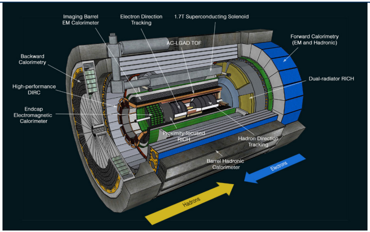

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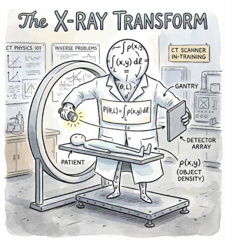

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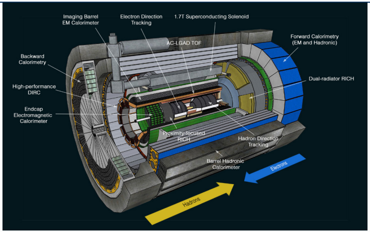

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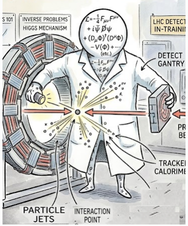

(d) Lepton-proton collision exp. $\equiv$???

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I want to construct and talk about an inverse problem with:

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- Overview (very very) briefly methods that have been used in the past.
- Describe the process of identifying these integral equations from the physics literature.
    - This was the heavy lifting of the paper
- Write the unified inverse problem, and inspect it.
- Construct a discretization algorithm for solution via regularization.

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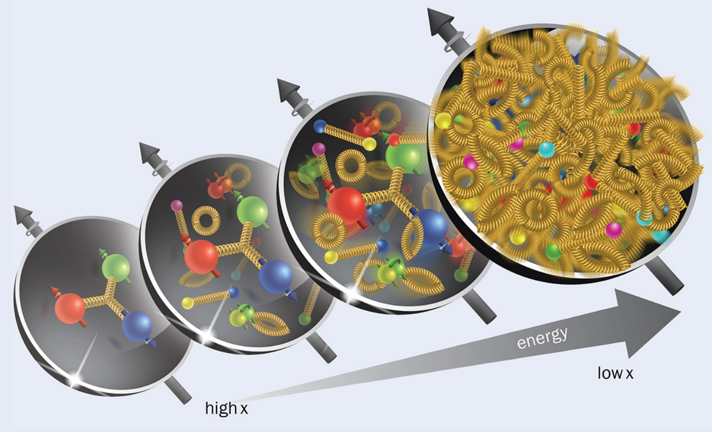
- Probably run out of time?

Proton is composed of quarks and gluons



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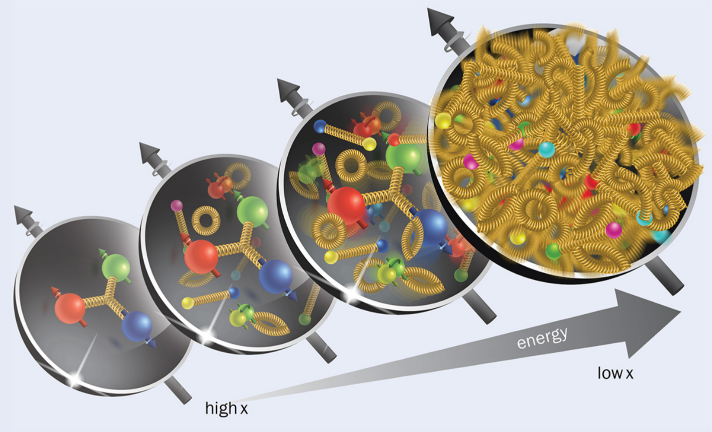

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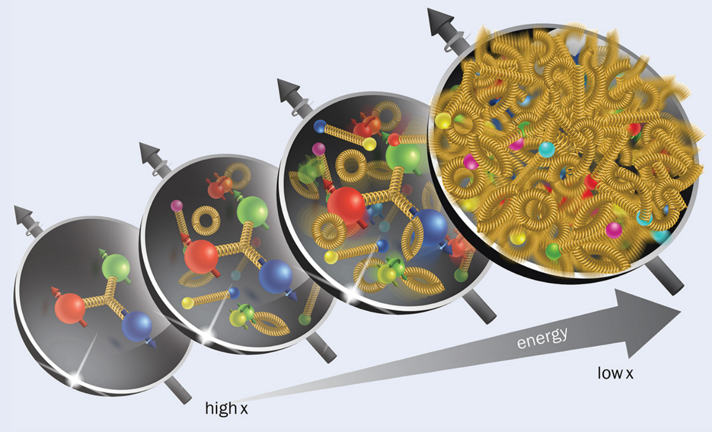

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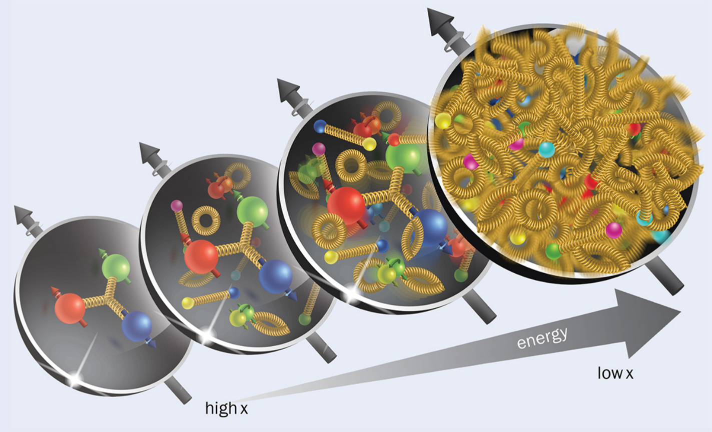

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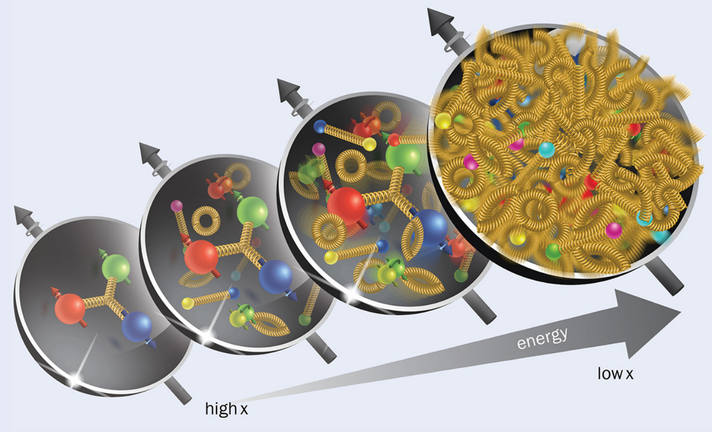

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We are going to formulate the problem of **inferring the distributions of all the quarks and the gluon**, and across all of the diverse regimes depicted.

Partons and their distribution functions in the proton

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- Quantum fluctuations in the structure produce so-called sea quarks that exist momentarily. (They can be probed in the experiment! PDFs for them too!)
- Particle collision experiments are inherently random by nature: the target is a quantum system, whose internal states we cannot precondition in the experiment. Basically, “we hit what we hit”, when a probe interacts with anything inside the proton.

Conservation laws as boundary constraints

The PDFs satisfy certain quantum field theoretic “conservation laws” known as sum rules:

- Total momentum fraction is conserved at all Q^2:

$$\int_0^1 \sum_{a\in\{g,d,u,..\}} x f_a(x,Q^2)\mathrm{d}x = \int_0^1 x\big[f_g(x,Q^2)+f_d(x,Q^2)+f_u(x,Q^2)+f_s(x,Q^2) \\ + f_c(x,Q^2)+f_b(x,Q^2)+f_t(x,Q^2) \\ + f_{\bar d}(x,Q^2)+f_{\bar u}(x,Q^2)+f_{\bar s}(x,Q^2) \\ + f_{\bar c}(x,Q^2)+f_{\bar b}(x,Q^2)+f_{\bar t}(x,Q^2)\big]\mathrm{d}x \equiv 1.$$

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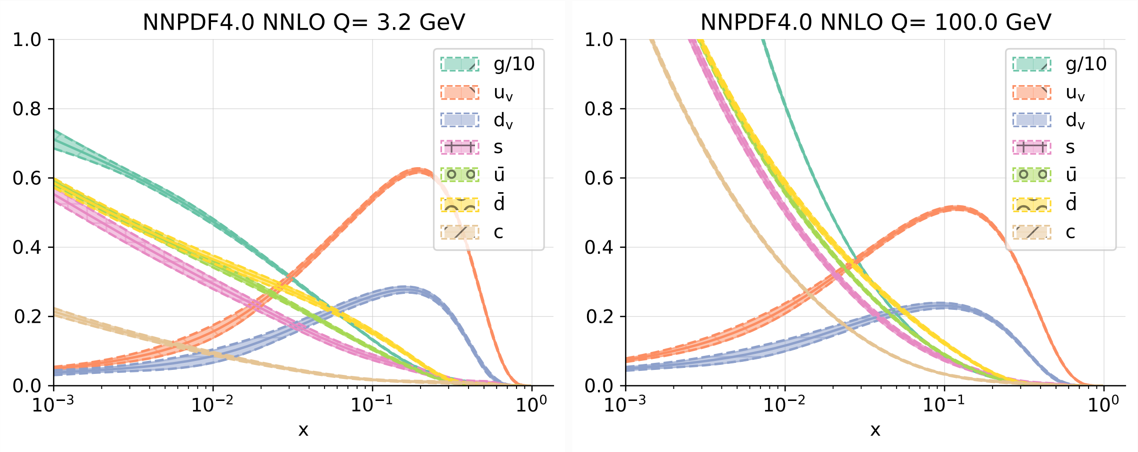
- Specific quantum numbers defined with respect to quark flavors are conserved: proton as a net number of 2 up quarks, and 1 down quark.

$$\int_0^1 f_u(x,Q^2) - f_{\bar{u}}(x,Q^2)\mathrm{d}x = N_u \equiv 2 \quad \forall\, Q^2 > 0.$$

State of the art of the extraction of the PDFs from data



NNPDF Collaboration, *Eur.Phys.J. C* **82**, 428 (2022)

Top 5 life hacks to know about the PDFs

- They are the most precise information we have about the structure of the proton:
  - Whether you're calibrating a particle collider, designing a new particle detector, studying quark gluon plasma, or searching for new elementary particles[1] to explain dark matter, you can (or need to) use the PDFs.

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- Since the 1980s they have been extracted from various diverse selections of data in various ways, which have used parameter fitting, or neural networks in current state of the art.
- What hasn't been done:
  - Proofs of existence or uniqueness of solutions.
  - What would be necessary number or optimal choice of measurements to use?
  - Solving them from data in a general sense has been considered impossible. (Without fitting a theoretically motivated model with parameters or neural network optimization)

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- Leading order (LO), Next-to-leading order (NLO): We look at QCD calculations done within a perturbative expansion. LO refers to the 0th term, and NLO to the 1st.
- Boson: elementary particle that acts as the carrier of an elementary force. We will see:
  - photon γ^* as the carrier of electromagnetic interaction
  - gluon g as the carrier the strong nuclear interaction
  - Z, W^+, and W^- bosons of the weak nuclear interaction.

Kinematic variables of high-energy scattering

The experimentally controllable kinematic variables of the scattering are:

- Bjorken-x, or just x: longitudinal momentum fraction carried by the parton struck in the interaction.
  - $\left(\frac{\text{parton mom. along beam axis}}{\text{proton mom. along beam axis}}\right)$
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$\Longrightarrow$ We are going to look at diverse probabilities of scattering as a function of these variables.

History of global analysis

Historically the functional parametrization ansatz for the PDFs has had variations of the form:

$$x f_a(x) \approx A x^{\alpha} (1-x)^{\beta} P(x), \quad A > 0,\ \alpha < 0,\ \beta > 0,$$

where a is the “flavor” index of the partons: g for gluon, d for down, and so on.

[2] *Eur.Phys.J. C* **82**, 428 (2022)

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Modern analyses such as those of the NNPDF collaboration[2] apply more advanced methods with Bayesian inference and neural networks.

- The goal of this work is to remove the reliance on the parametrized ansatz. (To enable testing it’s assumptions)

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The fundamental physical challenges of the problem

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- The inference of the PDFs is complicated by other unknowns, such as the precise masses of the quarks and the strength of the strong nuclear force, both of which depend on the momentum scale of the interaction, effects known as “running”.
- The general structure of the mathematical inverse problem doesn’t show up at the “easy” leading order. We are going directly after the general structure.

Setting up the foundation of the approach: the postulates

- **Postulate I**: The PDFs $f_a(x,Q^2)$, $a \in \{g, d, u, s, c, b, t, \bar{d}, \bar{u}, \bar{s}, \bar{c}, \bar{b}, \bar{t}\}$ are only meaningful as whole and must be solved simultaneously from data. We will construct a mathematical structure to treat the PDF family as a unified object.

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- **Postulate III**: Perturbative QCD and phenomenological description define a physically accurate problem of global analysis that can be tak....
  - TLDR: Perturbative QCD works and global analysis of the PDFs works
  - $\implies$ there is a mathematical inverse problem in there somewhere.

The additional simplifying assumptions

- **Assumption (i)** All other non-perturbative degrees of freedom than the proton PDFs are known be it either experimentally or phenomenologically. This includes parton masses, the running of the strong coupling $\alpha_s(\mu^2)$, and the nuclear modification factors $R^A(x, Q^2)$.
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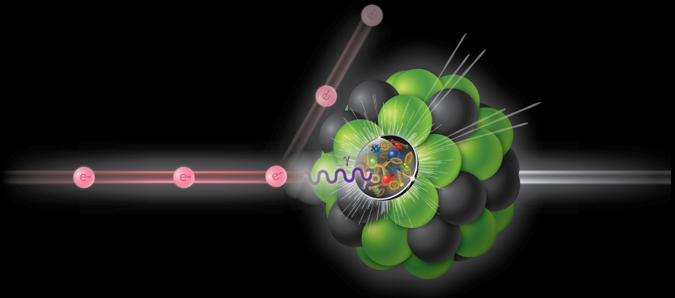
- **Assumption (ii)** A parton must be massless to be described by a PDF. Due to this, we mainly consider the theory of high-energy scattering for massless quarks.[...]
  - Parton has to be massless to be described by a PDF.
  - Heavy quarks (quarks with large masses: charm, bottom, top) are important to correctly describe real data.
  - We consider a toy-model of the practical inclusion of the heavy quarks in this inverse problem formulation.

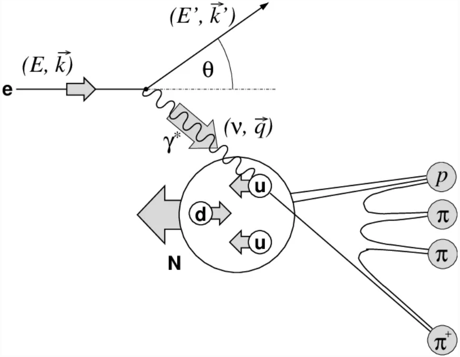
Deep inelastic scattering

- Proton and nucleus structures are studied in high-energy collisions.
- Electron as a point-like elementary particle is a nice clean probe to study the target
  - Interacts with the target via electromagnetic and weak nuclear forces.
  - Resolves elementary particles inside the target.



Flavors of deeply inelastic scattering

- An incoming lepton l scatters off the target proton P by exchanging a force carrier boson b:
  - The boson can be $b \in \{\gamma^*, Z, W^+, W^-\}$



Lepton–proton deeply inelastic scattering mediated by a photon γ^* (wavy line), which hits an up quark.

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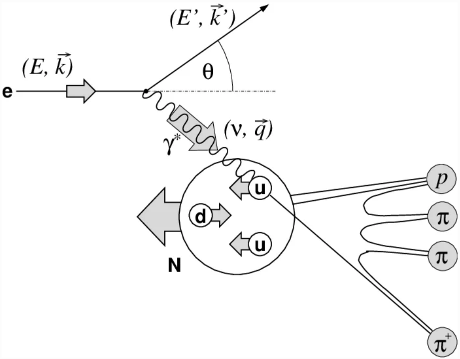
- The QCD theory prescription of the scattering interaction is dictated by the boson b exchanged:
  - photon γ^* can see electrically charged particles. (Quarks)
  - $Z, W^\pm$ bosons can see particles with weak charge. (Quarks)



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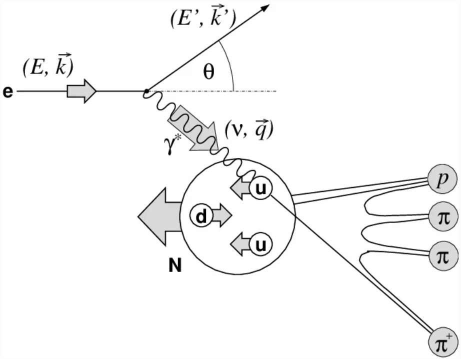
- The QCD theory prescription of the scattering interaction is dictated by the boson b exchanged:
  - photon γ^* can see electrically charged particles. (Quarks)
  - $Z, W^\pm$ bosons can see particles with weak charge. (Quarks)
- The proton contains gluons, how do we see those??
  - Gluons do not have electric or weak charge, so they cannot be directly probed!
  - The bosons *can* hit sea quarks, which precipitate from the gluon cloud. $\implies$ the gluon PDF has to be inferred indirectly.



Lepton–proton deeply inelastic scattering mediated by a photon γ^* (wavy line), which hits an up quark.

The DIS datasets that the real global analysis collaborations use

List of DIS experiment datasets that the global analyses use or consider.

- From *Eur.Phys.J. C* **82**, 428 (2022) by the NNPDF collaboration.

We are looking for the mathematical structure of these on a theoretical level.

Data set	Ref.	NNPDF3.1	NNPDF4.0	ABMP16	CT18	MSHT20
NMC F_2^d/F_2^p	[33]	✓	✓	✗	✗	✓
NMC $\sigma^{\mathrm{NC},p}$	[34]	✓	✓	✗	✓	✓
SLAC F_2^p, F_2^d	[35, 251]	✓	✓	✓	✗	✓
BCDMS F_2^p	[36]	✓	✓	✓	✓	✓
BCDMS F_2^d	[252]	✓	✓	✗	✓	✓
BCDMS, NMC, SLAC F_{L}	[34, 36, 251]	✗	✗	✗	✗	✓
CHORUS $\sigma_{CC}^{\nu}, \sigma_{CC}^{\bar{\nu}}$	[37]	✓	✓	✗	✗	✓
CHORUS	[253]	✗	✗	✓	✗	✗
NuTeV F_2, F_3	[254]	✗	✗	✗	✗	✓
NuTeV/CCFR $\sigma_{CC}^{\nu}, \sigma_{CC}^{\bar{\nu}}$	[38]	✓	✓	✓	✓	✓
EMC F_2^c	[44]	(✓)	(✓)	✗	✗	✗
NOMAD	[111]	✗	(✓)	✓	✗	✗
CCFR xF_3^p	[255]	✗	✗	✗	✓	✗
CCFR F_2^p	[256]	✗	✗	✗	✓	✗
CDSHW F_2^p, xF_3^p	[257]	✗	✗	✗	✓	✗
E665 F_2^p, F_2^d	[258]	✗	✗	✗	✗	✓
HERA NC, CC	[259]	✗	✗	✗	✗	✓
HERA I+II $\sigma_{\mathrm{NC,CC}}^{p}$	[40]	✓	✓	✓	✓	✗
HERA I+II $\sigma_{c\bar{c}}^{\mathrm{red}}$	[145]	✗	✓	✗	(✓)	✓
HERA I+II $\sigma_{b\bar{b}}^{\mathrm{red}}$	[145]	✗	✓	✗	(✓)	✗
HERA I+II $\sigma_{c\bar{c}}^{\mathrm{red}}$	[41]	✓	✗	✓	✓	✗
H1 $F_2^{c\bar{c}}$	[260]	✗	✗	✗	✓	✗
H1 $F_2^{b\bar{b}}$	[42]	✓	✗	✓	✗	✗
ZEUS $\sigma_{b\bar{b}}^{\mathrm{red}}$	[43]	✓	✗	✓	✗	✗
H1 F_{L}	[261]	✗	✗	✗	✓	✓
H1 and ZEUS F_{L}	[262, 263]	✗	✗	✗	✗	✓
ZEUS 820 (HQ) (1j)	[112]	✗	(✓)	✗	✗	✗
ZEUS 920 (HQ) (1j)	[113]	✗	(✓)	✗	✗	✗
H1 (LQ) (1j-2j)	[115]	✗	(✓)	✗	✗	✗
H1 (HQ) (1j-2j)	[116]	✗	(✓)	✗	✗	✗
ZEUS 920 (HQ) (2j)	[114]	✗	(✓)	✗	✗	✗

Table B.1. The fixed-target and collider DIS measurements used for PDF determination. For each PDF set, a blue tick indicates that the given dataset is included and a red cross that it is not included. A parenthesized tick denotes that a dataset was investigated but not included in the baseline fit.

Measurements considered in this work

Process	Parton	Observable	Experiment
$e^\pm p \to e^\pm X$	$q, \bar{q}, g$	$\sigma^{\pm}_{r,\mathrm{NC}}\,(\approx \sigma^{\gamma}_{r}),\ F_L^{\mathrm{NC}},\ xF_3^{\mathrm{NC}}$	HERA I+II, ZEUS
$e^+ p \to \bar{\nu} X$	$q, \bar{q}$	$\sigma^{\pm}_{r,\mathrm{CC}}$	HERA I+II
$e^\pm p \to e^\pm c\bar{c} X, e^\pm b\bar{b} X$	c, b, g	$\sigma_r^{c\bar{c}},\ \sigma_r^{b\bar{b}}$	HERA I+II
$\nu(\bar{\nu}) N \to \mu^-(\mu^+) X$	$q, \bar{q}$	$\sigma^{(\nu,\bar{\nu})N},\ F_2^{(\nu,\bar{\nu})N},\ xF_3^{(\nu,\bar{\nu})N}$	NuTeV, CHORUS
$ep \to eX, ed \to eX$	$q, \bar{q}, g$	F_L^{NC}	JLab
$\mu p \to X, \mu d \to X$	$q, \bar{q}$	Gottfried s.r. S_{G}, F_L^{NC}	NMC
$\nu(\bar{\nu}) N \to \mu^-(\mu^+) X$	$q, \bar{q}$	GLS s.r. S_{GLS}	CCFR/NuTeV
$\nu(\bar{\nu}) D \to \mu^-(\mu^+) X$	$q, \bar{q}$	Adler s.r. S_{A}	WA25 (BEBC)

Table: Particle scattering measurements considered in this work.

- I considered the inclusion of all of these as measureable constraints of the problem.
- We will have time for $\sim 3-5$? ¯_(ツ)_/¯
- N.B. Cross sections $\sigma \propto$ probability of scattering

Recognizing the universal shape: Electromagnetic DIS at NLO

The canonical pQCD result for the electromagnetic DIS cross section at NLO accuracy is[3]:

[3] G. Altarelli, R. K. Ellis and G. Martinelli, Nucl. Phys. B 143 (1978) 521.

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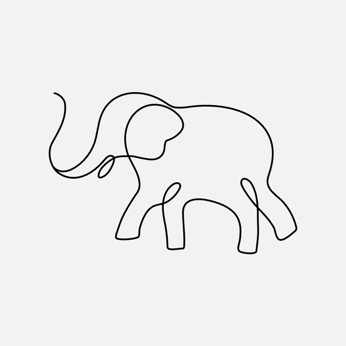

Figure: 22 pages of leading order physics not discussed.

[3] G. Altarelli, R. K. Ellis and G. Martinelli, Nucl. Phys. B 143 (1978) 521.

Recognizing the universal shape: Electromagnetic DIS at NLO

The canonical pQCD result for the electromagnetic DIS cross section at NLO accuracy is[3]:

$$
\begin{aligned}
\sigma_r^{\gamma}(x, Q^2, y) = &\sum_a e_a^2 x f_a(x, Q^2) \\
&+ \sum_a e_a^2 \int_x^1 x f_a\left(\frac{x}{z}, Q^2\right) \frac{\alpha_s}{4\pi} \left(c_{2,q}^{(1)}(z)_+ - \frac{y^2}{Y_+} c_{L,q}^{(1)}(z) \right) \frac{\mathrm{d}z}{z} \\
&+ \left(\sum_a e_a^2 \right) \int_x^1 x f_g\left(\frac{x}{z}, Q^2\right) \frac{\alpha_s}{4\pi} \left(c_{2,g}^{(1)}(z)_+ - \frac{y^2}{Y_+} c_{L,g}^{(1)}(z) \right) \frac{\mathrm{d}z}{z},
\end{aligned}
$$

where f_a are the PDFs to be solved, e_a are the fractional charges of the quarks $e_a \in \{\pm\frac{2}{3}, \pm\frac{1}{3}\}$, $\alpha_s \equiv \alpha_s(Q^2)$ is a scalar function that describes the strength of the strong interaction, and the kernel functions c_i are known as Wilson coefficient functions.

[3] G. Altarelli, R. K. Ellis and G. Martinelli, Nucl. Phys. B 143 (1978) 521.

Recognizing the universal shape: Electromagnetic DIS at NLO

Writing out the sum over parton flavors:

$$
\begin{aligned}
&\sigma_r^\gamma(x,Q^2,y) \equiv \\
&e_d^2 x f_d(x,Q^2) + e_u^2 x f_u(x,Q^2) + e_s^2 x f_s(x,Q^2) + e_c^2 x f_c(x,Q^2) + e_b^2 x f_b(x,Q^2) + e_t^2 x f_t(x,Q^2) \\
&+ e_{\bar d}^2 x f_{\bar d}(x,Q^2) + e_{\bar u}^2 x f_{\bar u}(x,Q^2) + e_{\bar s}^2 x f_{\bar s}(x,Q^2) + e_{\bar c}^2 x f_{\bar c}(x,Q^2) + e_{\bar b}^2 x f_{\bar b}(x,Q^2) + e_{\bar t}^2 x f_{\bar t}(x,Q^2) \\
&+ \int_x^1 \left[e_d^2 x f_d\left(\frac{x}{z},Q^2\right) + e_u^2 x f_u\left(\frac{x}{z},Q^2\right) + e_s^2 x f_s\left(\frac{x}{z},Q^2\right) + \cdots \right] \frac{\alpha_s}{4\pi} \left(c_{2,q}^{(1)}(z)_+ - \frac{y^2}{Y_+} c_{L,q}^{(1)}(z) \right) \frac{\mathrm{d}z}{z} \\
&+ \left(\sum_a e_a^2 \right) \int_x^1 x f_g\left(\frac{x}{z}, Q^2\right) \frac{\alpha_s}{4\pi} \left(c_{2,g}^{(1)}(z)_+ - \frac{y^2}{Y_+} c_{L,g}^{(1)}(z) \right) \frac{\mathrm{d}z}{z}.
\end{aligned}
$$

- Here we see the fundamental challenge of the problem: from this one observable it is *mathematically impossible* to separate uniquely any of the PDFs. This is the very reason the approach of global analysis, i.e. using all possible data simultaneously, was developed.

The shape of the inverse problem in a prototype equation

Consider the inference of two unknown functions $f(x)$ and $g(x)$ from two distinct measurements $b_1(q)$ and $b_2(q)$ that are only sensitive to the sum and difference of the functions $f(x)$ and $g(x)$:

$$b_1(q) \coloneqq \int K_1(x,q)(f(x)+g(x))\mathrm{d}x,$$
$$b_2(q) \coloneqq \int K_2(y,q)(f(y)-g(y))\mathrm{d}y.$$

We formalize the integrals as linear integral operators $\mathcal{K}_i$:

$$b_i(q) = \int K_i(x,q)(f(x)\pm g(x))\mathrm{d}x \eqqcolon \mathcal{K}_i(f\pm g) = \mathcal{K}_i(f) \pm \mathcal{K}_i(g).$$

The shape of the inverse problem in a prototype equation

Stacking the unknowns f, g and the observables b_i we get a matrix valued coupled problem:

$$\begin{bmatrix} \mathrm{b}_1 \\ \mathrm{b}_2 \end{bmatrix} = \begin{bmatrix} \mathcal{K}_1 & \mathcal{K}_1 \\ \mathcal{K}_2 & -\mathcal{K}_2 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}.$$

If the integral operators $\mathcal{K}_i$ have inverse operators $\mathcal{K}_i^{-1}$, the system has the explicit solution

$$f(x) = \frac{1}{2}\Big[(\mathcal{K}_1^{-1}b_1)(x) + (\mathcal{K}_2^{-1}b_2)(x)\Big]$$
$$g(x) = \frac{1}{2}\Big[(\mathcal{K}_1^{-1}b_1)(x) - (\mathcal{K}_2^{-1}b_2)(x)\Big].$$

The main point of the paper is to show that the problem of global analysis of the PDFs can be written into a coupled integral equation system of this form.

Forms of linear integral operators that appear

The full problem exhibits diverse versions of the following kinds of linear or integral equations:

$$
\begin{aligned}
O^{\sigma}_{\mathrm{LO}}(x, Q^2, y) &= L(x, Q^2, y)\mathbf{f}(x, Q^2),\\
O^{\sigma}_{\mathrm{1st}}(x, Q^2, y) &= \int_x^1 L(z, x, y, Q^2)\mathbf{f}(z, Q^2)\mathrm{d}z,\\
O^{\sigma}_{\mathrm{2nd}}(x, Q^2, y) &= \lambda(x, Q^2, y)\mathbf{f}(x, Q^2) + \int_x^1 L(z, x, y, Q^2)\mathbf{f}(z, Q^2)\mathrm{d}z,\\
O^{F}_{\mathrm{1st}}(x, Q^2) &= \int_x^1 L(z, x, Q^2)\mathbf{f}(z, Q^2)\mathrm{d}z,\\
O^{F}_{\mathrm{2nd}}(x, Q^2) &= \lambda(x, Q^2)\mathbf{f}(x, Q^2) + \int_x^1 L(z, x, Q^2)\mathbf{f}(z, Q^2)\mathrm{d}z,
\end{aligned}
$$

which we will keep in mind for when we construct the solution algorithm.

Writing out the canonical physics result in the new form

$$\sigma^{\gamma}_{r,\mathrm{NLO}}(x, Q^2, y) = \begin{bmatrix} \underset{\sim}{e}^2 \mathcal{I}_{2-L,g} & e_d^2 \mathcal{I}_{2-L,q} & \cdots & e_{\bar{t}}^2 \mathcal{I}_{2-L,q} \end{bmatrix} \begin{bmatrix} f_g(\cdot, Q^2) \\ f_d(\cdot, Q^2) \\ \vdots \\ f_{\bar{t}}(\cdot, Q^2) \end{bmatrix}$$

$$=: \left(\mathbf{A}^{\gamma}_{r,\mathrm{NLO}} \mathbf{f} \right)(x, Q^2, y),$$

where we defined:

$$\mathcal{I}_{2-L,q}[f](x, Q^2, y) := x \,\mathrm{Id} \circ f + \int_x^1 x f\left(x, Q^2\right) \frac{\alpha_s}{4\pi} \left(c^{(1)}_{2,q}\left(\frac{x}{z}\right)_+ - \frac{y^2}{Y_+} c^{(1)}_{L,q}\left(\frac{x}{z}\right) \right) \frac{\mathrm{d}z}{z}$$

$$\mathcal{I}_{2-L,g}[f](x, Q^2, y) := \int_x^1 x f\left(x, Q^2\right) \frac{\alpha_s}{4\pi} \left(c^{(1)}_{2,g}\left(\frac{x}{z}\right)_+ - \frac{y^2}{Y_+} c^{(1)}_{L,g}\left(\frac{x}{z}\right) \right) \frac{\mathrm{d}z}{z}.$$

With these the sum over parton flavors becomes an inner product of a row vector of linear integral operators with the parton distribution tensor, i.e. PDFs stacked in a vector.

Identifying the shape in other observables at NLO

- That was electromagnetic DIS at NLO accuracy

Identifying the shape in other observables at NLO

- That was electromagnetic DIS at NLO accuracy
- We still need:
  - Neutral current DIS at NLO (γ and Z boson exchange)
  - Charged current DIS at NLO ($W^{\pm}$ boson exchange)
  - Heavy quark production in DIS at NLO (charm / bottom can be tagged in experiment)
  - Neutrino–nucleus DIS at NLO ($W^{\pm}$ boson exchange, but with a neutrino beam)
  - Total momentum conservation sum rule
  - Flavor number conservation sum rule
  - Flavor asymmetry sum rules (measured): Adler, Gottfried, Gross–Llewellyn-Smith.
    - “Need” in the sense that I included these in the paper, but we “need” to skip them for time.

Neutral current DIS

Neutral current is the more general picture of the pure EM interaction:

- in addition to pure γ there are Z and γZ contributions.

The neutral current reduced cross section in $e^{\pm} + p \to e^{\pm} + X$ scattering is[4]:

$$\sigma^{\pm}_{r,\mathrm{NC}}(x, Q^2, y) = F_2^{\pm,\mathrm{NC}}(x, Q^2) \mp \frac{Y_-}{Y_+} x F_3^{\pm,\mathrm{NC}}(x, Q^2) - \frac{y^2}{Y_+} F_L^{\pm,\mathrm{NC}}(x, Q^2),$$

where the structure functions F_i are in general of the form:

$$F_i(x, Q^2) := \left(\sum_a C_i^a \otimes f_a \right)(x, Q^2),$$

where the Wilson coefficient functions C_i^a depend on the physics of the observable, and

$$(C \otimes f)(x) := \int_x^1 C(y) f\left(\frac{x}{y}\right) \frac{\mathrm{d}y}{y}.$$

[4] H1, ZEUS collaboration, Abramowicz, H., Abt, I., Adamczyk, L. *et al.* Eur. Phys. J. C 75, 580 (2015).

Neutral current forward operator at NLO

$$\sigma^{\pm}_{r,\mathrm{NC}}(x, Q^2, y) = \begin{bmatrix} \phi^{\pm}_{g}\left(\mathcal{C}_{2,g} - \frac{y^2}{Y_+}\mathcal{C}_{L,g}\right) & \mathcal{C}^{\mathrm{NC}}_{q,d} & \cdots & \mathcal{C}^{\mathrm{NC}}_{q,\bar{t}} \end{bmatrix} \begin{bmatrix} f_g(\cdot, Q^2) \\ f_d(\cdot, Q^2) \\ \vdots \\ f_{\bar{t}}(\cdot, Q^2) \end{bmatrix}$$

$$=: \left(\mathbf{A}^{\pm}_{r,\mathrm{NC}}\mathbf{f}\right)(x, Q^2, y),$$

where we defined the combined integral operator for the quarks:

$$\mathcal{C}^{\mathrm{NC}}_{q,a}[f] := \phi^{\pm}_{2,a}\left(\mathcal{C}_{2,q}[f] - \frac{y^2}{Y_+}\mathcal{C}_{L,q}[f]\right) \mp \frac{xY_-}{Y_+}\phi^{\pm}_{3,a}(-1)^{\delta_{a\bar{a}}}\mathcal{C}_{3,q}[f],$$

where the helicity-averaged effective squared parity-preserving electro-weak charge $\phi^{\pm}_{2,a}$ is:

$$\phi^{\pm}_{2,a} = e_a^2 - 2e^a g^a_V \left(g^{e,\pm}_V\right)\eta_{\gamma Z}(Q^2) + \left((g^a_V)^2 + (g^a_A)^2\right)\left(\left(g^{e,\pm}_V\right)^2 + \left(g^{e,\pm}_A\right)^2\right)\eta_Z(Q^2),$$

and the effective parity-breaking counterpart of Eq. $\phi^{\pm}_{3,a}$:

$$\phi^{\pm}_{3,a} = -2e^a g^a_A g^{e,\pm}_A \eta_{\gamma Z}(Q^2) + 2g^a_V g^a_A \left(2g^{e,\pm}_V g^{e,\pm}_A\right)\eta_Z(Q^2).$$

Charged current DIS

Charged current DIS is the process $e^-(e^+) + p \to \nu(\bar{\nu}) + X$.

- Electron / positron traveling almost at the speed of light interacts with the proton and **vanishes!**
- Mediated by W^+, W^- bosons. The lepton turns into a neutrino which is not detected.

The experimentally measured reduced charged current cross section is[5]

$$\sigma^{\pm}_{r,\mathrm{CC}}(x, Q^2, y) = \frac{Y_+}{2} F_2^{W^\pm}(x, Q^2) \mp \frac{Y_-}{2} x F_3^{W^\pm}(x, Q^2) - \frac{y^2}{2} F_L^{W^\pm}(x, Q^2).$$

The charged current structure functions $F_i^{W^\pm}$ encode the physics that

- W^+ only interacts with “down” and “anti-up” flavors $(d, s, b, \bar{u}, \bar{c}, \bar{t})$
- W^- interacts with their counterparts $(u, c, t, \bar{d}, \bar{s}, \bar{b})$

$\therefore$ CC gives new separation power to distinguish between flavors! (Has been a part of PDF extraction for ages, we want to incorporate in this inverse problems formalism.)

[5] H1, ZEUS collaboration, Abramowicz, H., Abt, I., Adamczyk, L. *et al.* Eur. Phys. J. C 75, 580 (2015).

Charged current DIS forward operators at NLO

$$\sigma_{r,\mathrm{CC}}^{\pm,\mathrm{NLO}}(x,Q^2,y) = \left(\mathbf{A}_{r,\mathrm{CC}}^{\pm,\mathrm{NLO}}\mathbf{f}\right)(x,Q^2,y),$$

where

$$\mathbf{A}_{r,\mathrm{CC}}^{+}(x,Q^2,y) = \left[\tfrac{n_f Y_+}{2}\left(\mathcal{C}_{2,g} - \tfrac{y^2}{Y_+}\mathcal{C}_{L,g}\right)\ \mathcal{C}_{q,d}^{W^+}\ \ 0\ \ \mathcal{C}_{q,s}^{W^+}\ \ 0\ \ \mathcal{C}_{q,b}^{W^+}\ \ 0\ \ 0\ \ \mathcal{C}_{q,\bar{u}}^{W^+}\ \ 0\ \ \mathcal{C}_{q,\bar{c}}^{W^+}\ \ 0\ \ \mathcal{C}_{q,\bar{t}}^{W^+}\right],$$

$$\mathbf{A}_{r,\mathrm{CC}}^{-}(x,Q^2,y) = \left[\tfrac{n_f Y_+}{2}\left(\mathcal{C}_{2,g} - \tfrac{y^2}{Y_+}\mathcal{C}_{L,g}\right)\ 0\ \ \mathcal{C}_{q,u}^{W^-}\ \ 0\ \ \mathcal{C}_{q,c}^{W^-}\ \ 0\ \ \mathcal{C}_{q,t}^{W^-}\ \ \mathcal{C}_{q,\bar{d}}^{W^-}\ \ 0\ \ \mathcal{C}_{q,\bar{s}}^{W^-}\ \ 0\ \ \mathcal{C}_{q,\bar{b}}^{W^-}\ \ 0\right],$$

where we defined the integral operator:

$$\begin{aligned}\mathcal{C}_{q,a}^{W^\pm}[f] &:= \frac{Y_+}{2}\mathcal{C}_{2,q}[f] \mp (-1)^{\delta_{a\bar{a}}}\frac{Y_-}{2}x\mathcal{C}_{3,q}[f] - \frac{y^2}{2}\mathcal{C}_{L,q}[f] \\ &\equiv \int_x^1 \left(\frac{Y_+}{2}c_{2,q}\left(\frac{x}{z}\right) \mp (-1)^{\delta_{a\bar{a}}}\frac{Y_-}{2}xc_{3,q}\left(\frac{x}{z}\right) - \frac{y^2}{2}c_{L,q}\left(\frac{x}{z}\right)\right) f(z)\frac{\mathrm{d}z}{z}.\end{aligned}$$

Heavy flavor production in DIS

Charm and bottom quarks are so heavy that their production can be tagged in experiment[6]:

$$e^{\pm} + p \to e^{\pm} + c\bar{c}\,(b\bar{b}) + X.$$

Their contributions to the structure functions are of the form

$$\begin{aligned} F_i^{Q\overline{Q}}(x, n_f\!+\!1, Q^2, m^2) = \sum_{k=1}^{n_f} e_k^2 L_{i,q}^{\mathsf{NS}}\left(x, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \left[f_k(x,\mu^2,n_f) + f_{\overline{k}}(x,\mu^2,n_f)\right] \\ + e_Q^2 \Bigg[H_{i,q}^{\mathsf{PS}}\left(x, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \Sigma(x,\mu^2,n_f) \\ + H_{i,g}^{\mathsf{S}}\left(x, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes f_g(x,\mu^2,n_f) \Bigg], \quad i \in \{2, L\}. \end{aligned}$$

[6] H1, ZEUS collaboration, H. Abramowicz *et al.* Eur. Phys. J. C 78, 473 (2018).

Charm production forward operator with heavy quark effects

$$F_i^{c\bar{c}}(x, n_f + 1 = 4, Q^2, m_c^2) = \left(\mathbf{B}_{i,\mathrm{NLO}}^{c\bar{c}}\mathbf{f}\right)(x, Q^2, m_c)$$

$$:= \begin{bmatrix} \mathcal{H}_{i,g,c}^{\mathsf{S}} & \mathcal{I}_{i,d,c}^{L+H} & \mathcal{I}_{i,u,c}^{L+H} & \mathcal{I}_{i,s,c}^{L+H} & 0 & 0 & 0 & \mathcal{I}_{i,\bar{d},c}^{L+H} & \mathcal{I}_{i,\bar{u},c}^{L+H} & \mathcal{I}_{i,\bar{s},c}^{L+H} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_g(\cdot, Q^2) \\ f_d(\cdot, Q^2) \\ \vdots \\ f_{\bar{t}}(\cdot, Q^2) \end{bmatrix}$$

where

$$\mathcal{H}_{i,g,Q}^{\mathsf{S}}[f](x, Q^2, m^2) := e_Q^2 \int_{ax}^{1} H_{i,g}^{\mathsf{S}}\left(\frac{x}{z}, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) f(z, Q^2)\frac{\mathrm{d}z}{z},$$

$$\mathcal{I}_{i,q,Q}^{L+H}[f](x, Q^2, m^2) := \int_{ax}^{1} \left\{ e_q^2 L_{i,q}^{\mathsf{NS}}\left(\frac{x}{z}, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) + e_Q^2 H_{i,q}^{\mathsf{PS}}\left(\frac{x}{z}, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \right\} f(z, Q^2)\frac{\mathrm{d}z}{z}.$$

Neutrino DIS at NLO

- Neutrinos ν are notoriously difficult to detect $\implies$ Need big dense detectors and huge flux
- Detectors of made of lead, iron, argon, carbon, water *are not made of free protons*
- Need to generalize the formalism for nuclear target, this is composed of Z protons and $A-Z$ neutrons. $\implies$ iso-scalar mixing.

Up and down quark PDFs are mixed as:

$$\begin{bmatrix} C'_d \\ C'_u \end{bmatrix} = \frac{1}{A} \begin{bmatrix} Z & A-Z \\ A-Z & Z \end{bmatrix} \begin{bmatrix} C_d \\ C_u \end{bmatrix} := \mathbf{w}_{A,Z} \begin{bmatrix} C_d \\ C_u \end{bmatrix},$$

which affects the full flavor basis as:

$$\mathbf{C}' = \begin{bmatrix} 1 & & & & \\ & \mathbf{w}_{A,Z} & & & \\ & & \mathbf{I}^{4\times 4} & & \\ & & & \mathbf{w}_{A,Z} & \\ & & & & \mathbf{I}^{4\times 4} \end{bmatrix} \mathbf{C} =: \mathbf{W}_{A,Z}\mathbf{C}.$$

Neutrino DIS forward operator at NLO

The forward operator for a nuclear target becomes:

$$\mathbf{W}_{A,Z}\mathbf{A}^{\nu(\bar{\nu})p}_{r,\mathrm{NLO}}$$
$$= \begin{bmatrix} \mathcal{C}_g^{\nu(\bar{\nu})p} & \mathcal{C}_d^{\nu(\bar{\nu})^A_Z\mathrm{N}} & \mathcal{C}_u^{\nu(\bar{\nu})^A_Z\mathrm{N}} & \mathcal{C}_s^{\nu(\bar{\nu})p} & \cdots & \mathcal{C}_t^{\nu(\bar{\nu})p} & \mathcal{C}_{\bar{d}}^{\nu(\bar{\nu})^A_Z\mathrm{N}} & \mathcal{C}_{\bar{u}}^{\nu(\bar{\nu})^A_Z\mathrm{N}} & \mathcal{C}_{\bar{s}}^{\nu(\bar{\nu})p} & \cdots & \mathcal{C}_{\bar{t}}^{\nu(\bar{\nu})p} \end{bmatrix}$$

where we use the short-hands:

$$\mathcal{C}_g^{\nu(\bar{\nu})p}[f] := \frac{Y_+}{2}\left(\mathcal{C}_{2,g}[f] - \frac{y^2}{Y_+}\mathcal{C}_{L,g}[f]\right),$$
$$\mathcal{C}_{d(\bar{d})}^{\nu(\bar{\nu})^A_Z\mathrm{N}}[f] := \frac{Z}{A}\mathcal{C}_{q,d(\bar{d})}^{\nu(\bar{\nu})p}[f] + \frac{A-Z}{A}\mathcal{C}_{q,u(\bar{u})}^{\nu(\bar{\nu})p}[f],$$
$$\mathcal{C}_{u(\bar{u})}^{\nu(\bar{\nu})^A_Z\mathrm{N}}[f] := \frac{A-Z}{A}\mathcal{C}_{q,d(\bar{d})}^{\nu(\bar{\nu})p}[f] + \frac{Z}{A}\mathcal{C}_{q,u(\bar{u})}^{\nu(\bar{\nu})p}[f],$$
$$\mathcal{C}_q^{\nu(\bar{\nu})p}[f] := \mathcal{C}_{q,a}^{\nu(\bar{\nu})p}[f].$$

Identifying the shape in QCD sum rules

- Total momentum is conserved:

$$\int_0^1 \sum_{a\in\{g,d,u,..\}} x f_a(x,Q^2)\mathrm{d}x = \int_0^1 x\big[f_g(x,Q^2) + f_d(x,Q^2) + f_u(x,Q^2) + f_s(x,Q^2) \\ + f_c(x,Q^2) + f_b(x,Q^2) + f_t(x,Q^2) \\ + f_{\bar{d}}(x,Q^2) + f_{\bar{u}}(x,Q^2) + f_{\bar{s}}(x,Q^2) \\ + f_{\bar{c}}(x,Q^2) + f_{\bar{b}}(x,Q^2) + f_{\bar{t}}(x,Q^2)\big]\mathrm{d}x \equiv 1.$$

Define

$$\mathcal{I}^{(x)}[f] := \int_0^1 x f(x)\mathrm{d}x,$$

with which the total momentum conservation sum rule becomes:

$$\mathcal{I}_{\text{t.m.c.}}[\mathbf{f}] := \left[\mathcal{I}^{(x)} \quad \mathcal{I}^{(x)} \quad \mathcal{I}^{(x)} \quad \mathcal{I}^{(x)} \quad \mathcal{I}^{(x)} \quad \mathcal{I}^{(x)} \quad \mathcal{I}^{(x)} \quad \mathcal{I}^{(x)} \quad \mathcal{I}^{(x)} \quad \mathcal{I}^{(x)} \quad \mathcal{I}^{(x)} \quad \mathcal{I}^{(x)} \quad \mathcal{I}^{(x)}\right] \mathbf{f}.$$

Formulating the inverse problem of global analysis

- We have written many specific observables into linear forward operators.
- The remaining step is to formalize the inverse problem of global analysis.

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- We want to state a mathematical problem where the individually intractable inverse problems must be solved simultaneously to gain mathematical power to resolve the PDFs.

Formulating the inverse problem of global analysis

- We have written many specific observables into linear forward operators.
- The remaining step is to formalize the inverse problem of global analysis.
- We want to state a mathematical problem where the individually intractable inverse problems must be solved simultaneously to gain mathematical power to resolve the PDFs.
- This is naturally expressed as a coupled system of inverse problems.

Collecting the forward operators into a unified problem

$$\mathbf{O}^{\mathrm{WD}} := \begin{bmatrix} \mathbf{O}^{\sigma} \\ \mathbf{O}^{F} \\ \mathbf{O}^{S} \end{bmatrix} = \int \mathbf{G}_{\mathrm{NLO}}^{lp\to l'X} \mathbf{f}$$

is the coupled problem, where for example the cross section observable block is:

$$\mathbf{O}^{\sigma} := \begin{bmatrix} \sigma_r^{\gamma}(x,Q^2,y) \\ \sigma_{r,\mathrm{NC}}^{+}(x,Q^2,y) \\ \sigma_{r,\mathrm{NC}}^{-}(x,Q^2,y) \\ \sigma_{r,\mathrm{CC}}^{+}(x,Q^2,y) \\ \sigma_{r,\mathrm{CC}}^{-}(x,Q^2,y) \\ \sigma_r^{c\bar{c}}(x,Q^2,y) \\ \sigma_r^{b\bar{b}}(x,Q^2,y) \\ \sigma_r^{\nu {}^A_Z\mathrm{N}}(x,Q^2,y) \\ \sigma_r^{\bar{\nu} {}^A_Z\mathrm{N}}(x,Q^2,y) \end{bmatrix} = \int \begin{bmatrix} \mathbf{A}_r^{\gamma}(x,Q^2,y) \\ \mathbf{A}_{r,\mathrm{NC}}^{+}(x,Q^2,y) \\ \mathbf{A}_{r,\mathrm{NC}}^{-}(x,Q^2,y) \\ \mathbf{A}_{r,\mathrm{CC}}^{+}(x,Q^2,y) \\ \mathbf{A}_{r,\mathrm{CC}}^{-}(x,Q^2,y) \\ \mathbf{A}_r^{c\bar{c}}(x,Q^2,y) \\ \mathbf{A}_r^{b\bar{b}}(x,Q^2,y) \\ \mathbf{A}_r^{\nu(\bar{\nu}) {}^A_Z\mathrm{N}}(x,Q^2,y) \\ \mathbf{A}_r^{\nu(\bar{\nu}) {}^A_Z\mathrm{N}}(x,Q^2,y) \end{bmatrix} \mathbf{f}(\cdot,Q^2).$$

Collecting the forward operators into a unified problem

$$\mathbf{O}^{\mathrm{WD}} := \begin{bmatrix} \mathbf{O}^{\sigma} \\ \mathbf{O}^{F} \\ \mathbf{O}^{S} \end{bmatrix} = \int \mathbf{G}_{\mathrm{NLO}}^{lp\to l'X} \mathbf{f},$$

and the structure function block is:

$$\mathbf{O}^{F} := \begin{bmatrix} F_L^{\gamma}(x,Q^2) \\ xF_3^{\mathrm{NC}}(x,Q^2) \\ F_2^{\nu N}(x,Q^2) \\ xF_3^{\nu N}(x,Q^2) \\ F_2^{\bar{\nu} N}(x,Q^2) \\ xF_3^{\bar{\nu} N}(x,Q^2) \end{bmatrix} = \int \begin{bmatrix} \mathbf{B}_L^{\gamma}(x,Q^2) \\ \mathbf{B}_{3,\mathrm{NC}}^{\pm}(x,Q^2) \\ \mathbf{B}_2^{\nu_Z^A \mathrm{N}}(x) \\ \mathbf{B}_3^{\nu_Z^A \mathrm{N}}(x) \\ \mathbf{B}_2^{\bar{\nu}_Z^A \mathrm{N}}(x) \\ \mathbf{B}_3^{\bar{\nu}_Z^A \mathrm{N}}(x) \end{bmatrix} \mathbf{f}(\cdot,Q^2)$$

Collecting the forward operators into a unified problem

$$\mathbf{O}^{\mathrm{WD}} := \begin{bmatrix} \mathbf{O}^{\sigma} \\ \mathbf{O}^{F} \\ \mathbf{O}^{S} \end{bmatrix} = \int \mathbf{G}_{\mathrm{NLO}}^{lp\to l'X} \mathbf{f},$$

and the sum rule constraint block is:

$$\mathbf{O}^{S} := \begin{bmatrix} S_{\mathrm{t.m.c.}}(Q^2) \\ S_{\mathrm{f.n.c.}}(Q^2) \\ S_{\mathrm{GLS}}(Q^2) \\ S_{\mathrm{A}}(Q^2) \\ S_{\mathrm{G}}(Q^2) \end{bmatrix} = \int \begin{bmatrix} \mathcal{I}_{\mathrm{t.m.c.}} \\ \mathcal{I}_{\mathrm{f.n.c.}} \\ \mathcal{I}_{\mathrm{GLS}} \\ \mathcal{I}_{\mathrm{A}} \\ \mathcal{I}_{\mathrm{G}} \end{bmatrix} \mathbf{f}(\cdot, Q^2).$$

Each element in the kernel matrix is a row-form integral operator we had to rewrite from the established physics results, which by-construction reproduce the original QCD results.

The expanding to see some structure of the inverse problem

The mathematical shape of global analysis is a huge coupled system of integral equations:

$$\mathbf{O}^{\mathrm{WD}} = \int_0^1 \mathbf{G}^{lp\to l'X}_{\mathrm{NLO}}(z,x,Q^2,y)\mathbf{f}(z,Q^2)\frac{\mathrm{d}z}{z}$$

$$\Leftrightarrow \begin{bmatrix} \sigma_r^\gamma(x,Q^2,y) \\ \vdots \\ F_L^\gamma(x,Q^2) \\ \vdots \\ S_{\mathrm{t.m.c.}}(Q^2) \\ \vdots \end{bmatrix} = \int_0^1 \begin{bmatrix} \underset{\sim}{e}^2\mathcal{I}_{2-L,g}\theta_x & e_d^2\mathcal{I}_{2-L,q}\theta_x & \cdots & e_{\bar{t}}^2\mathcal{I}_{2-L,q}\theta_x \\ \vdots & \ddots & \cdots & \vdots \\ \underset{\sim}{e}^2\mathcal{I}_{L,g}\theta_x & e_d^2\mathcal{I}_{L,q}\theta_x & \cdots & e_{\bar{t}}^2\mathcal{I}_{L,q}\theta_x \\ \vdots & \ddots & \cdots & \vdots \\ z^2 & z^2 & \cdots & z^2 \\ \vdots & \ddots & \cdots & \vdots \end{bmatrix} \begin{bmatrix} f_g(z,Q^2) \\ f_d(z,Q^2) \\ \vdots \\ f_{\bar{t}}(z,Q^2) \end{bmatrix} \frac{\mathrm{d}z}{z}.$$

This system-level integral equation structure has not been identified previously!?

- Global analysis of the PDFs has always been a large number of formulae to compute and compare against data. Here we see a substantial amount of structure.

The 325 integral operators (DIS world data integral equation system)

$$\mathbf{O}^{\mathrm{WD},lp\to l'X}(x,Q^2,y) =$$

$$
\begin{bmatrix}
e^2\mathcal{I}_{2-L,g} & e_d^2\mathcal{I}_{2-L,q} & e_u^2\mathcal{I}_{2-L,q} & e_s^2\mathcal{I}_{2-L,q} & e_c^2\mathcal{I}_{2-L,q} & e_b^2\mathcal{I}_{2-L,q} & e_t^2\mathcal{I}_{2-L,q} & e_{\bar d}^2\mathcal{I}_{2-L,q} & e_{\bar u}^2\mathcal{I}_{2-L,q} & e_{\bar s}^2\mathcal{I}_{2-L,q} & e_{\bar c}^2\mathcal{I}_{2-L,q} & e_{\bar b}^2\mathcal{I}_{2-L,q} & e_{\bar t}^2\mathcal{I}_{2-L,q} \\
\phi_g^+\mathcal{C}_{2-L,g} & \mathcal{C}^{+,\mathrm{NC}}_{q,d} & \mathcal{C}^{+,\mathrm{NC}}_{q,u} & \mathcal{C}^{+,\mathrm{NC}}_{q,s} & \mathcal{C}^{+,\mathrm{NC}}_{q,c} & \mathcal{C}^{+,\mathrm{NC}}_{q,b} & \mathcal{C}^{+,\mathrm{NC}}_{q,t} & \mathcal{C}^{+,\mathrm{NC}}_{q,\bar d} & \mathcal{C}^{+,\mathrm{NC}}_{q,\bar u} & \mathcal{C}^{+,\mathrm{NC}}_{q,\bar s} & \mathcal{C}^{+,\mathrm{NC}}_{q,\bar c} & \mathcal{C}^{+,\mathrm{NC}}_{q,\bar b} & \mathcal{C}^{+,\mathrm{NC}}_{q,\bar t} \\
\phi_g^-\mathcal{C}_{2-L,g} & \mathcal{C}^{-,\mathrm{NC}}_{q,d} & \mathcal{C}^{-,\mathrm{NC}}_{q,u} & \mathcal{C}^{-,\mathrm{NC}}_{q,s} & \mathcal{C}^{-,\mathrm{NC}}_{q,c} & \mathcal{C}^{-,\mathrm{NC}}_{q,b} & \mathcal{C}^{-,\mathrm{NC}}_{q,t} & \mathcal{C}^{-,\mathrm{NC}}_{q,\bar d} & \mathcal{C}^{-,\mathrm{NC}}_{q,\bar u} & \mathcal{C}^{-,\mathrm{NC}}_{q,\bar s} & \mathcal{C}^{-,\mathrm{NC}}_{q,\bar c} & \mathcal{C}^{-,\mathrm{NC}}_{q,\bar b} & \mathcal{C}^{-,\mathrm{NC}}_{q,\bar t} \\
\frac{Y_+}{2}\mathcal{C}_{2-L,g} & \mathcal{C}^{W^+}_{q,d} & 0 & \mathcal{C}^{W^+}_{q,s} & 0 & \mathcal{C}^{W^+}_{q,b} & 0 & 0 & \mathcal{C}^{W^+}_{q,\bar u} & 0 & \mathcal{C}^{W^+}_{q,\bar c} & 0 & \mathcal{C}^{W^+}_{q,\bar t} \\
\frac{Y_+}{2}\mathcal{C}_{2-L,g} & 0 & \mathcal{C}^{W^-}_{q,u} & 0 & \mathcal{C}^{W^-}_{q,c} & 0 & \mathcal{C}^{W^-}_{q,t} & \mathcal{C}^{W^-}_{q,\bar d} & 0 & \mathcal{C}^{W^-}_{q,\bar s} & 0 & \mathcal{C}^{W^-}_{q,\bar b} & 0 \\
\mathcal{H}^{\mathrm{S}}_{r,g,c} & \mathcal{I}^{L+H}_{r,d,c} & \mathcal{I}^{L+H}_{r,u,c} & \mathcal{I}^{L+H}_{r,s,c} & 0 & 0 & 0 & \mathcal{I}^{L+H}_{r,\bar d,c} & \mathcal{I}^{L+H}_{r,\bar u,c} & \mathcal{I}^{L+H}_{r,\bar s,c} & 0 & 0 & 0 \\
\mathcal{H}^{\mathrm{S}}_{r,g,b} & \mathcal{I}^{L+H}_{r,d,b} & \mathcal{I}^{L+H}_{r,u,b} & \mathcal{I}^{L+H}_{r,s,b} & 0 & 0 & 0 & \mathcal{I}^{L+H}_{r,\bar d,b} & \mathcal{I}^{L+H}_{r,\bar u,b} & \mathcal{I}^{L+H}_{r,\bar s,b} & 0 & 0 & 0 \\
\mathcal{C}_g^{\nu p} & \mathcal{C}_d^{\nu_Z^A\mathrm{N}} & \mathcal{C}_u^{\nu_Z^A\mathrm{N}} & \mathcal{C}_s^{\nu p} & \mathcal{C}_c^{\nu p} & \mathcal{C}_b^{\nu p} & \mathcal{C}_t^{\nu p} & \mathcal{C}_{\bar d}^{\nu_Z^A\mathrm{N}} & \mathcal{C}_{\bar u}^{\nu_Z^A\mathrm{N}} & \mathcal{C}_{\bar s}^{\nu p} & \mathcal{C}_{\bar c}^{\nu p} & \mathcal{C}_{\bar b}^{\nu p} & \mathcal{C}_{\bar t}^{\nu p} \\
\mathcal{C}_g^{\bar\nu p} & \mathcal{C}_d^{\bar\nu_Z^A\mathrm{N}} & \mathcal{C}_u^{\bar\nu_Z^A\mathrm{N}} & \mathcal{C}_s^{\bar\nu p} & \mathcal{C}_c^{\bar\nu p} & \mathcal{C}_b^{\bar\nu p} & \mathcal{C}_t^{\bar\nu p} & \mathcal{C}_{\bar d}^{\bar\nu_Z^A\mathrm{N}} & \mathcal{C}_{\bar u}^{\bar\nu_Z^A\mathrm{N}} & \mathcal{C}_{\bar s}^{\bar\nu p} & \mathcal{C}_{\bar c}^{\bar\nu p} & \mathcal{C}_{\bar b}^{\bar\nu p} & \mathcal{C}_{\bar t}^{\bar\nu p} \\
2\frac{\alpha_s}{\pi}e^2\mathcal{I}_{L,g} & e_d^2x\mathcal{I}_{L,q} & e_u^2x\mathcal{I}_{L,q} & e_s^2x\mathcal{I}_{L,q} & e_c^2x\mathcal{I}_{L,q} & e_b^2x\mathcal{I}_{L,q} & e_t^2x\mathcal{I}_{L,q} & e_{\bar b}^2x\mathcal{I}_{L,q} & e_{\bar u}^2x\mathcal{I}_{L,q} & e_{\bar s}^2x\mathcal{I}_{L,q} & e_{\bar c}^2x\mathcal{I}_{L,q} & e_{\bar b}^2x\mathcal{I}_{L,q} & e_{\bar t}^2x\mathcal{I}_{L,q} \\
0 & \phi^\pm_{3,d}\mathcal{C}_{3,q} & \phi^\pm_{3,u}\mathcal{C}_{3,q} & \phi^\pm_{3,s}\mathcal{C}_{3,q} & \phi^\pm_{3,c}\mathcal{C}_{3,q} & \phi^\pm_{3,b}\mathcal{C}_{3,q} & \phi^\pm_{3,t}\mathcal{C}_{3,q} & -\phi^\pm_{3,\bar d}\mathcal{C}_{3,q} & -\phi^\pm_{3,\bar u}\mathcal{C}_{3,q} & -\phi^\pm_{3,\bar s}\mathcal{C}_{3,q} & -\phi^\pm_{3,\bar c}\mathcal{C}_{3,q} & -\phi^\pm_{3,\bar b}\mathcal{C}_{3,q} & -\phi^\pm_{3,\bar t}\mathcal{C}_{3,q} \\
\frac{1}{2}\mathcal{C}_{2,g} & \frac{Z}{A}\mathcal{C}_{2,q} & \frac{A-Z}{A}\mathcal{C}_{2,q} & \mathcal{C}_{2,q} & 0 & \mathcal{C}_{2,q} & 0 & \frac{A-Z}{A}\mathcal{C}_{2,q} & \frac{Z}{A}\mathcal{C}_{2,q} & 0 & \mathcal{C}_{2,q} & 0 & \mathcal{C}_{2,q} \\
0 & \frac{Z}{A}\mathcal{C}_{3,q} & \frac{A-Z}{A}\mathcal{C}_{3,q} & \mathcal{C}_{3,q} & 0 & \mathcal{C}_{3,q} & 0 & -\frac{A-Z}{A}\mathcal{C}_{3,q} & -\frac{Z}{A}\mathcal{C}_{3,q} & 0 & -\mathcal{C}_{3,q} & 0 & -\mathcal{C}_{3,q} \\
\frac{1}{2}\mathcal{C}_{2,g} & \frac{A-Z}{A}\mathcal{C}_{2,q} & \frac{Z}{A}\mathcal{C}_{2,q} & 0 & \mathcal{C}_{2,q} & 0 & \mathcal{C}_{2,q} & \frac{Z}{A}\mathcal{C}_{2,q} & \frac{A-Z}{A}\mathcal{C}_{2,q} & \mathcal{C}_{2,q} & 0 & \mathcal{C}_{2,q} & 0 \\
0 & \frac{A-Z}{A}\mathcal{C}_{3,q} & \frac{Z}{A}\mathcal{C}_{3,q} & 0 & \mathcal{C}_{3,q} & 0 & \mathcal{C}_{3,q} & -\frac{Z}{A}\mathcal{C}_{3,q} & -\frac{A-Z}{A}\mathcal{C}_{3,q} & -\mathcal{C}_{3,q} & 0 & -\mathcal{C}_{3,q} & 0 \\
0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} \\
\mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} \\
0 & \mathcal{I}^{(1)} & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & -\mathcal{I}^{(1)} & 0 & 0 & 0 & 0 \\
0 & -\mathcal{I}^{(1)} & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & \mathcal{I}^{(1)} & -\mathcal{I}^{(1)} & 0 & 0 & 0 & 0 \\
0 & -\mathcal{I}^{(1)} & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
f_g(\cdot,Q^2) \\ f_d(\cdot,Q^2) \\ f_u(\cdot,Q^2) \\ f_s(\cdot,Q^2) \\ f_c(\cdot,Q^2) \\ f_b(\cdot,Q^2) \\ f_t(\cdot,Q^2) \\ f_{\bar d}(\cdot,Q^2) \\ f_{\bar u}(\cdot,Q^2) \\ f_{\bar s}(\cdot,Q^2) \\ f_{\bar c}(\cdot,Q^2) \\ f_{\bar b}(\cdot,Q^2) \\ f_{\bar t}(\cdot,Q^2)
\end{bmatrix}
$$

How many observables are needed?

Number of necessary or best observables not conclusively settled even for DIS.

Data set	Ref.	NNPDF3.1	NNPDF4.0	ABMP16	CT18	MSHT20
NMC F_2^d/F_2^p	[33]	✓	✓	✗	✗	✓
NMC $\sigma^{\mathrm{NC},p}$	[34]	✓	✓	✗	✓	✓
SLAC F_2^p, F_2^d	[35, 251]	✓	✓	✓	✗	✓
BCDMS F_2^p	[36]	✓	✓	✓	✓	✓
BCDMS F_2^d	[252]	✓	✓	✗	✓	✓
BCDMS, NMC, SLAC F_{L}	[34, 36, 251]	✗	✗	✗	✗	✓
CHORUS $\sigma_{CC}^{\nu}, \sigma_{CC}^{\bar{\nu}}$	[37]	✓	✓	✗	✗	✓
CHORUS	[253]	✗	✗	✓	✗	✗
NuTeV F_2, F_3	[254]	✗	✗	✗	✗	✓
NuTeV/CCFR $\sigma_{CC}^{\nu}, \sigma_{CC}^{\bar{\nu}}$	[38]	✓	✓	✓	✓	✓
EMC F_2^c	[44]	(✓)	(✓)	✗	✗	✗
NOMAD	[111]	✗	(✓)	✓	✗	✗
CCFR xF_3^p	[255]	✗	✗	✗	✓	✗
CCFR F_2^p	[256]	✗	✗	✗	✓	✗
CDSHW F_2^p, xF_3^p	[257]	✗	✗	✗	✓	✗
E665 F_2^p, F_2^d	[258]	✗	✗	✗	✗	✓
HERA NC, CC	[259]	✗	✗	✗	✗	✓
HERA I+II $\sigma_{\mathrm{NC,CC}}^p$	[40]	✓	✓	✓	✓	✗
HERA I+II $\sigma_{c\bar{c}}^{\mathrm{red}}$	[145]	✗	✓	✗	(✓)	✓
HERA I+II $\sigma_{b\bar{b}}^{\mathrm{red}}$	[145]	✗	✓	✗	(✓)	✗
HERA I+II $\sigma_{c\bar{c}}^{\mathrm{red}}$	[41]	✓	✗	✓	✓	✗
H1 $F_2^{c\bar{c}}$	[260]	✗	✗	✗	✓	✗
H1 $F_2^{b\bar{b}}$	[42]	✓	✗	✓	✗	✗
ZEUS $\sigma_{b\bar{b}}^{\mathrm{red}}$	[43]	✓	✗	✓	✗	✗
H1 F_{L}	[261]	✗	✗	✗	✓	✓
H1 and ZEUS F_{L}	[262, 263]	✗	✗	✗	✗	✓
ZEUS 820 (HQ) (1j)	[112]	✗	(✓)	✗	✗	✗
ZEUS 920 (HQ) (1j)	[113]	✗	(✓)	✗	✗	✗
H1 (LQ) (1j-2j)	[115]	✗	(✓)	✗	✗	✗
H1 (HQ) (1j-2j)	[116]	✗	(✓)	✗	✗	✗
ZEUS 920 (HQ) (2j)	[114]	✗	(✓)	✗	✗	✗

Table B.1. The fixed-target and collider DIS measurements used for PDF determination. For each PDF set, a blue tick indicates that the given dataset is included and a red cross that it is not included. A parenthesized tick denotes that a dataset was investigated but not included in the baseline fit.

Number of observables and the rank of the forward operator

In this formulation we can consider the rank of the coupled system of integral equations generated by the world data forward operator $\mathbf{G}_{\mathrm{NLO}}^{lp\to l'X}(z, x, Q^2, y)$.

- 13 unknown PDFs would require a minimum of 13 integral equation constraints in $\mathbf{G}$?

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- What would be the optimal choice for the most separation between parton flavors?
- How to quantify “linear independence” of the integral operator rows?

Number of observables and the rank of the forward operator

In this formulation we can consider the rank of the coupled system of integral equations generated by the world data forward operator $\mathbf{G}_{\mathrm{NLO}}^{lp\to l'X}(z, x, Q^2, y)$.

- 13 unknown PDFs would require a minimum of 13 integral equation constraints in $\mathbf{G}$?
- What would be the optimal choice for the most separation between parton flavors?
- How to quantify “linear independence” of the integral operator rows?
- What happens when the number of operator rows is larger than 13? (Standard practice in global analysis of the PDFs: does the problem become over determined? Strong constraint on the self-consistency of all the row operators.)

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Open questions: What are the conditions for the full coupled integral equation problem to have any solutions? Or a unique solution?

Physics constraint for the solutions from QCD

Perturbative QCD predicts the Q^2-dependence of the PDFs:

$$\frac{\partial}{\partial \log Q^2} f_i(x, Q^2) = \frac{\alpha_s(Q^2)}{4\pi} \sum_j^{n_f} \int_x^1 P_{ij}(z, Q) f_j\left(\frac{z}{x}, Q^2\right) \frac{\mathrm{d}z}{z}, \qquad \text{(DGLAP)}$$

where the kernel functions P_{ij} depend on the partons i and j.

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where the kernel functions P_{ij} depend on the partons i and j. This can be (and has been) written in a matrix form:

$$\frac{\partial}{\partial \log Q^2} \mathbf{f}(x, Q^2) = \frac{\partial}{\partial \log Q^2} \begin{bmatrix} f_g(x, Q^2) \\ f_d(x, Q^2) \\ \vdots \\ f_{\bar{t}}(x, Q^2) \end{bmatrix} = \begin{bmatrix} \mathcal{P}_{gg} & \mathcal{P}_{gq} & \cdots & \mathcal{P}_{gq} \\ \mathcal{P}_{qg} & \mathcal{P}_{qq} & \cdots & \mathcal{P}_{qq} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{P}_{qg} & \mathcal{P}_{qq} & \cdots & \mathcal{P}_{qq} \end{bmatrix} \begin{bmatrix} f_g(\cdot, Q^2) \\ f_d(\cdot, Q^2) \\ \vdots \\ f_{\bar{t}}(\cdot, Q^2) \end{bmatrix} =: \mathbf{P}_{\mathrm{DGLAP}} \mathbf{f}.$$

- Conventionally built-in as a hard requirement in PDF fits.
  - PDFs parametrized at scale Q_0^2, solved with DGLAP for $Q^2 \geq Q_0^2$, and compared with data.
- Pleasing duality between DGLAP and the new coupled system of integral equations.

Flavor number schemes describe broad regimes of data

- The world data goes from all the way from low energy to blisteringly high energy.
  - Low energy in the sense that there's not enough energy to produce the heavy quarks. ($E = mc^2$, kinetic energy is turned into mass.)
  - High in the sense that all parton masses become negligible[7]: $Q^2 \gg m_a^2$ for all flavors.

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  - As Q^2 is raised, inert heavy flavors "heat up" and begin to be produced: their masses are quantitatively significant in the forward problem.
  - At very high Q^2, $Q^2 \gg m_a^2$, the mass of the parton becomes negligible compared to the other dynamics of the scattering, and it appears to behave similarly as the other approximately massless quarks. Engages with the QCD dynamics between flavors.

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The most simple flavor number scheme

- Theory frameworks to describe this multi-scale physics in Q^2 are known as flavor number schemes.
- This work heavily leans on the most simple yet workable such framework: zero-mass variable flavor number scheme (ZM-VFNS):

$$O^{\mathrm{ZM-VFNS}}(Q^2) := \begin{cases} O^{(3)}(Q^2) & \text{if } Q^2 < m_c^2 \\ O^{(4)}(Q^2) & \text{if } m_c^2 \leq Q^2 < m_b^2 \\ O^{(5)}(Q^2) & \text{if } m_b^2 \leq Q^2 < m_t^2 \\ O^{(6)}(Q^2) & \text{if } m_t^2 \leq Q^2, \end{cases}$$

  - Quarks always treated as massless.
  - Q^2 of data dictates which quarks are thought to contribute to the measurement.
  - Observable O is defined piece-wise on intervals in Q^2.

A toy model FNS to include some heavy flavor effects

In this work I also consider the toy-model of a FNS with some heavy flavor effects:

$$O^{\mathrm{3cb+ZM-VFNS}}(Q^2) := \begin{cases} O^{\mathrm{FFNS3cb}}(Q^2) & \text{if } Q^2 < m_b^2 \\ O^{(5)}(Q^2) & \text{if } m_b^2 \leq Q^2 < m_t^2 \\ O^{(6)}(Q^2) & \text{if } m_t^2 \leq Q^2, \end{cases}$$

which is only changed at the lowest interval by introducing the heavy quark mass effects in the charm and bottom production observables.

The *real* full inverse problem with a flavor number scheme

With the flavor number scheme, the integral equation system of global analysis becomes:

$$\mathbf{O}^{\mathrm{WD}}_{\mathrm{FNS}} := \begin{bmatrix} \mathbf{O}^{\mathrm{WD}}_{Q^2<m_b^2} \\ \mathbf{O}^{\mathrm{WD}}_{m_b^2\leq Q^2<m_t^2} \\ \mathbf{O}^{\mathrm{WD}}_{m_t^2\leq Q^2} \end{bmatrix} = \int \begin{bmatrix} \mathbf{G}^{lp\to l'X}_{Q^2<m_b^2} \\ \mathbf{G}^{lp\to l'X}_{m_b^2\leq Q^2<m_t^2} \\ \mathbf{G}^{lp\to l'X}_{m_t^2\leq Q^2} \end{bmatrix} \mathbf{f} =: \int \mathbf{G}^{lp\to l'X}_{\mathrm{FNS}} \mathbf{f}.$$

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The functional optimization problem then becomes with the DGLAP PDE constraint:

$$\underset{\mathbf{f},\mathbf{y}}{\text{minimize}} \left\| \mathbf{O}^{\mathrm{WD}}_{\mathrm{FNS}} - \int \mathbf{G}^{lp\to l'X}_{\mathrm{FNS}} \mathbf{f} \right\|^2_{L^2(\Omega)} + \lambda \|\mathbf{y}\|^2_{L^2(\Omega)},$$

subject to $\mathbf{y} := \dfrac{\partial}{\partial \log Q^2} \mathbf{f}(x, Q^2) - \mathbf{P}^{\mathrm{DGLAP}}_{\mathrm{FNS}} \mathbf{f}.$

Generality of the formalism

Any observable that satisfies the collinear factorization of pQCD:

$$F_i^b(x,Q^2) = \sum_a \int_x^1 C_{i,a}^b\left(\frac{x}{z},\frac{Q^2}{\mu^2},\alpha_s(\mu^2)\right) f_{a,H}(z,\mu^2)\frac{\mathrm{d}z}{z} + O(1\,\mathrm{GeV}/Q)$$

is amenable to this approach, and since the perturbative corrections go to C_i, this approach is at least theoretically possible also at higher orders (NNLO, N^3LO,..).

- State of the art of PDF extraction is at N^3LO accuracy.
  - Corrections up to α_s^3 included in the perturbative expansion.
- General result for DIS type observables.

On the solution of the parton level integral equations

Wiener–Hopf integral equations can be of first or second-kind:

$$\int_0^\infty k(x-y)f(y)\mathrm{d}y = g(x), \quad x \in (0,\infty),$$
$$f(x) - \int_0^\infty k(x-y)f(y)\mathrm{d}y = g(x), \quad x \in (0,\infty),$$

where $k(x-y)$ is known as a difference kernel. Conditions for the existence of (unique) solutions are known in many cases[8].

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where $k(x-y)$ is known as a difference kernel. Conditions for the existence of (unique) solutions are known in many cases[8]. The Wilson coefficient convolutions can be rewritten:

$$(C \otimes f)(x) := \int_x^1 C\left(\frac{x}{z}\right) f(z) \frac{\mathrm{d}z}{z} \equiv -\int_0^{\ln(x)} \tilde{C}(\tilde{x}-\tilde{z})\tilde{f}(\tilde{z})\mathrm{d}\tilde{z},$$

which would appear to be quite closely related to the Wiener–Hopf equations. $\tilde{C}$ can be a plus-distribution (i.e. not in L^1), so inspection of applicability of existing results needed.

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New(?) features in a mathematical inverse problem

- 13 unknown functions which cannot be directly or independently accessed via any means.

[9] The NNPDF Collaboration. Evidence for intrinsic charm quarks in the proton, *Nature* **608**, 483–487 (2022)

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- The formalism has to leverage $N \geq 13$ diverse experimental constraints and sum rules, which form different kinds of integral equations. Wiener-Hopf of 1st & 2nd kinds, and more.

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- Integral kernels of some constraints exhibit plus-distributions.
- Number of unknowns also depends on the energy scale.
  - PDE constraint that changes between momentum/energy regimes. (n_f)
  - $c, b, t, \bar{c}, \bar{b}, \bar{t}$ can only be produced if there is enough kinetic energy available to transform into mass. $(E = mc^2)$
  - Charm is a bit more complex matter as it might exist inside the proton "intrinsically"[9], but for extrinsic production the above applies.

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Discretizing the problem

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A key technique is the so-called stacking form, which reduces higher dimensional linear multiplication into regular matrix-vector product. We will need to use this for:

- parton flavors
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- Q^2

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- parton flavors
- integration over z
- Q^2

Further stacking is done for datapoints in y, and for different observables to form a discretized formulation of the full world data problem.

Stacking form

Starting from an observable of the form: $O_a(x) = \int_x^1 C\left(\frac{x}{z}\right) f_a(z) \frac{\mathrm{d}z}{z}$

1. Nyström quadrature method: Discretization in z and x.
   - Define quadrature $\mathcal{Q}_n := \sum_{j=1}^n \omega_j \frac{f(z_j)}{z_j} \approx \int_x^1 f(z) \frac{\mathrm{d}z}{z}$
   - Convolution by quadrature $(\mathcal{C}_n f)(x) := \mathcal{Q}_n\left(C\left(\frac{x}{\cdot}\right) f\right) := \sum_{j=1}^n \omega_j \frac{C\left(\frac{x}{z_j}\right)}{z_j} f(z_j)$
   - Discretize both in z and x: $M_{jk} := \omega_j \frac{C\left(\frac{x_k}{z_j}\right)}{z_j}, \quad 1 \le j \le n, \quad 1 \le k \le m$
   - Observable values on discretized nodes x_k by matrix multiplication: $\underset{\sim}{Q}_a := (O_a(x_1), \ldots, O_a(x_m))^T = M \underset{\sim}{f}_a$

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3. Discretization in Q^2. (Repeat the above steps for each node Q_l^2.)
   - Stack: $\mathbf{f}_N := \begin{bmatrix} \mathbf{f}_n(Q_1^2) & \mathbf{f}_n(Q_2^2) & \cdots & \mathbf{f}_n(Q_N^2) \end{bmatrix}^T$
   - $\mathbf{A}_{a,\nu ml} := A_a(z_\nu, x_m, Q_l^2)$; forward operator is discretized in z, x, Q^2.
   - $O(x_m, Q_l^2) = \mathbf{A}_{ml} \mathbf{f}_n(Q_l^2) = \sum_a \sum_\nu A_{a,\nu ml} f_a(z_\nu, Q_l^2)$.

Stacking form 2 – discretization boogaloo

4. Discretization in y. (Repeat the previous steps for each node in y.)
   - $O(x, Q^2, y) = \begin{bmatrix} O_{MN}(y_1) \\ \vdots \\ O_{MN}(y_K) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{MN}(y_1) \\ \vdots \\ \mathbf{A}_{MN}(y_K) \end{bmatrix} \mathbf{f}_N =: \mathbf{A}_{MNK}\mathbf{f}_N,$

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   - $O_a(x) = \int_x^1 C\left(\frac{x}{z}\right) \int_x^1 \left(\delta(z'-z) - \frac{x}{z}\delta(z'-x)\right) f(z')\mathrm{d}z' \frac{\mathrm{d}z}{z} - f(x)c(x)$
   - $\int_x^1 \left(\delta(z'-z) - \frac{x}{z}\delta(z'-x)\right) f(z')\mathrm{d}z' \approx \left(\delta_{z'_{j'}, z_j} - \frac{x_m}{z_j}\delta_{z'_{j'}, x_m}\right) f(z'_{j'}).$

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The discretized problem becomes a matrix$\times$vector form linear inverse problem:

- $\begin{bmatrix} \sigma_r^\gamma(x_m,Q_n^2,y_k)_{1\le m\le M,\,1\le n\le N,\,1\le k\le K} \\ F_L^\gamma(x_m,Q_n^2)_{1\le m\le M,\,1\le n\le N} \\ S_{\text{t.m.c}}(Q_n^2)_{1\le n\le N} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^\gamma_{r,\text{NLO},(MNK)} \\ \mathbf{B}^\gamma_{L,(MN)} \\ \mathbf{Y}^{\text{t.m.c.}}_{n,N} \end{bmatrix} \mathbf{f}_N$

Discussion and outlook

- I formulated the problem of extracting the PDFs from DIS data into a mathematical inverse problem
  - Needed to recognize the high level structure present in diverse set of DIS observables.
  - Reviewed the DIS observables at LO and NLO accuracy, and wrote them in the new inverse problem formalism.
  - Formalized an inverse problem that needs to use arbitrarily many measurement modalities.

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  - Inverse theory for the integral equations: what can we say about existence and uniqueness of solutions? At the parton level? How about at the full coupled problem level?
  - Is the Wiener-Hopf framing useful?

Discussion and outlook

- I formulated the problem of extracting the PDFs from DIS data into a mathematical inverse problem
  - Needed to recognize the high level structure present in diverse set of DIS observables.
  - Reviewed the DIS observables at LO and NLO accuracy, and wrote them in the new inverse problem formalism.
  - Formalized an inverse problem that needs to use arbitrarily many measurement modalities.
- Constructed a proof of principle discretization method for computational reconstruction method for the problem.
- Opens a path for further investigation:
  - What is the optimal number of observables, and what are they?
  - Can the forward operator be proven to have full rank with sufficiently many observable and sum rule constraints? (Might depend on the perturbative accuracy used?)
  - Inverse theory for the integral equations: what can we say about existence and uniqueness of solutions? At the parton level? How about at the full coupled problem level?
  - Is the Wiener-Hopf framing useful?
  - A more elegant solution method? (We constructed a brute force discretization reliant on regularization methods)

Discussion and outlook

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  - Parametrization-bias-free extraction of the PDFs: mathematical path towards “indirect sensing”?

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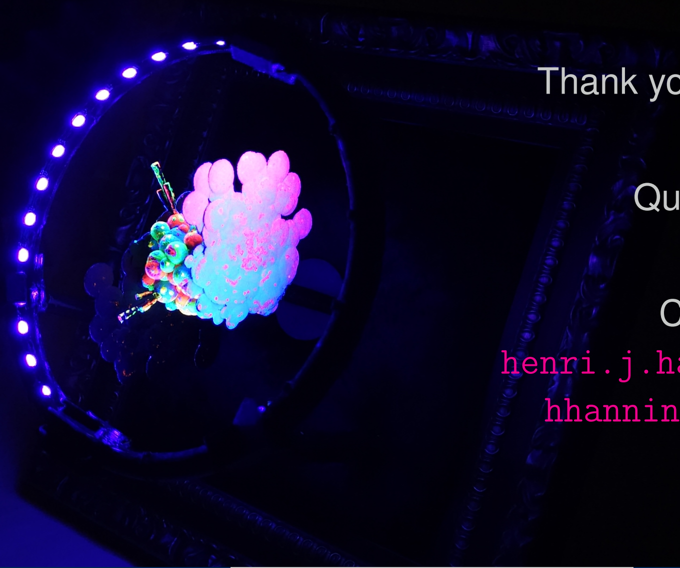
- Is the “parton distribution tensor” physically significant?
  - Was constructed purely as a mathematical convenience, but has some similarity with the notion of energy–momentum tensor in general relativity, which is a tensor-field that describes the density and flux of energy and momentum in spacetime.
  - Reflects the fact that proton is a composite hadron described by the hadronic tensor $W^{\mu\nu}$?



Thank you for listening!

Questions?

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Appendix

Plus-distributions

The Wilson coefficient functions acting as kernels can contain plus-distributions, which are defined as:

$$\int_0^1 a(z)_+ f(z)\mathrm{d}z := \int_0^1 a(x)\left(f(z) - f(1)\right).$$

Writing this out for the convolutions for the ratio of the arguments in the interval $[x, 1]$:

$$\begin{aligned}(c_+ \otimes f)(x) &:= \int_x^1 c_+(z) f\left(\frac{x}{z}\right)\frac{\mathrm{d}z}{z} \\ &= \int_x^1 C\left(\frac{x}{z}\right)\left(f(z) - \frac{x}{z}f(x)\right)\frac{\mathrm{d}z}{z} - f(x)\int_0^x C(z)\mathrm{d}z \\ &=: \int_x^1 C\left(\frac{x}{z}\right)\int_x^1 \left(\delta(z'-z) - \frac{x}{z}\delta(z'-x)\right) f(z')\mathrm{d}z'\frac{\mathrm{d}z}{z} - f(x)c(x).\end{aligned}$$

These are used to cancel out divergences between terms that appear when integrating $\int_x^{1-\varepsilon} c_+(z)\mathrm{d}z \underset{\varepsilon\to 0}{\to} \infty$.