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Mathematical inverse problem for the world data inference of the parton distribution functions of the proton

Center of Excellence in Quark Matter seminar

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October 1, 2026

Preprint arXiv:2609.11896 [hep-ph]
Under review in JHEP

Special thanks to Felix for discussions and correspondence!

How to solve the PDFs from DIS world data in a functional sense?

- PDF extraction schematically:

- 1 Parametrize PDFs at initial scale Q_0^2 : $f_a(x, Q_0^2)$
- 2 Evolve to $Q^2 > Q_0^2$ with DGLAP evolution equation. $f_a(x, Q^2)$ for $Q^2 \geq Q_0^2$.
- 3 Compute observables from the PDFs $f_a(x, Q^2)$
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- In the new paper I set out to identify mathematical structure in global analysis and pQCD, and based on those observations constructed a new mathematical inverse problem.
 - Guiding idea: no ansatz functional form, no parameter fitting, whatever the cost.
 - The cost would seem to be new mathematics in inverse problems, integral equations.

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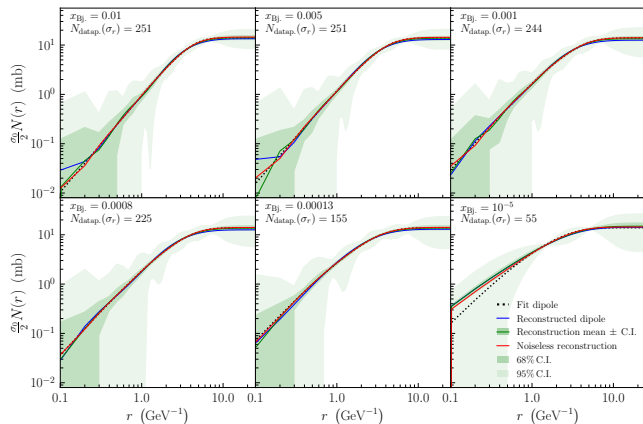
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 - Non-linear and **notoriously** ill-posed, and yet it can work in real world applications!
 - Low resolution but real time, and only needs small electrode pads on subject.
 - (Electrical conductivity can be medical imaging $\implies$ global analysis can image the PDFs.)

Functional linearity

The dipole integral approach is leveraging the fact that the measurement σ_r is linearly dependent on the unknown function:

$$O(\alpha f + \beta g) = \alpha O(f) + \beta O(g), \quad \forall \alpha, \beta \in \mathbb{R}.$$

or perhaps more transparently:

$$O(f) = O\left(\frac{1}{2}f + \frac{1}{2}f\right) = \frac{1}{2}O(f) + \frac{1}{2}O(f).$$

We are going to leverage this linearity in the mathematical formulation of global analysis.

- This limits the scope to lepton–hadron DIS and sum rules in this initial formulation.

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What do the PDFs look like?

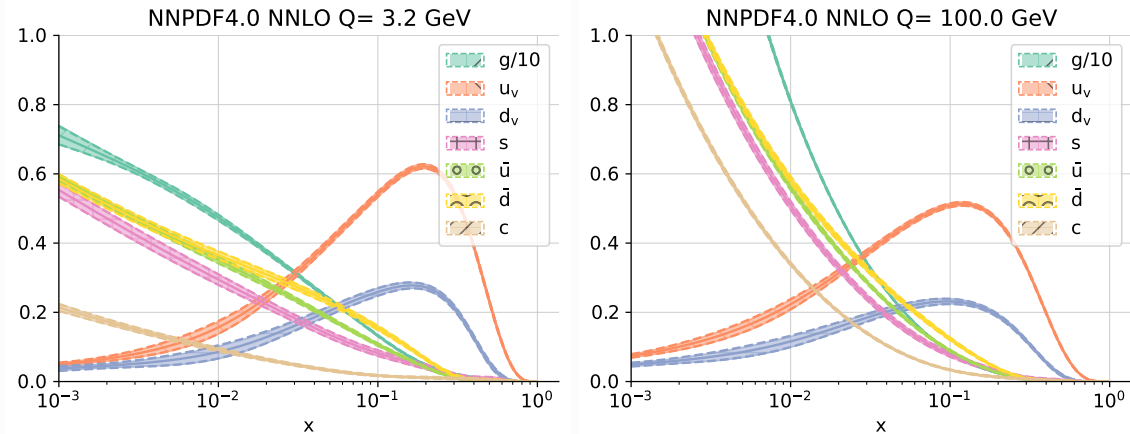


Figure: NNPDF Collaboration, *Eur.Phys.J. C* **82**, 428 (2022)

- PDFs are differentiable in x , given DGLAP also in Q^2 , and integrable due to sum rules.
- These are the functional properties we seek to leverage.

The fundamental physical challenges of the problem

Any extraction of the PDFs has to address a number of substantial hurdles:

- Fundamental inaccessibility. Free quarks or gluons cannot be observed, so the structure of bound states like the proton need to be studied.

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- The inference of the PDFs is complicated by other unknowns, such as the precise masses of the quarks and $\alpha_s(Q^2)$. I assume these to be known experimentally or phenomenologically.
- The general structure of the mathematical inverse problem doesn't show up at the “easy” leading order. We are going directly after the general convolution structure from collinear factorization.

Setting up the foundation of the approach: the postulates

- **Postulate I:** The PDFs $f_a(x, Q^2)$, $a \in \{g, d, u, s, c, b, t, \bar{d}, \bar{u}, \bar{s}, \bar{c}, \bar{b}, \bar{t}\}$ are only meaningful as whole and must be solved simultaneously from data. We will construct a mathematical structure to treat the PDF family as a unified object.

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- **Postulate II:** The PDFs $f_a(x, Q^2)$ defined after renormalization are sufficiently smooth and integrable functions $f_a(x, Q^2)$ to be solved from data in a functional sense. The experimental data are measurements of theoretically smooth quantities such as scattering cross sections, and the mathematical problem is to solve the smooth PDFs from a set of smooth observables.
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 - This is a generalization of the approach of defining an ansatz functional parametrization used in fitting the PDFs.
- **Postulate III:** Perturbative QCD and phenomenological description define a physically accurate problem of global analysis that can be tak....
 - TLDR: Perturbative QCD works and global analysis of the PDFs works
 - $\implies$ there is a mathematical inverse problem in there somewhere.

The additional simplifying assumptions

- **Assumption (i)** All other non-perturbative degrees of freedom than the proton PDFs are known be it either experimentally or phenomenologically. This includes parton masses, the running of the strong coupling $\alpha_s(\mu^2)$, and the nuclear modification factors $R^A(x, Q^2)$.
 - Read: anything else non-trivial than the PDFs are assumed to be known.

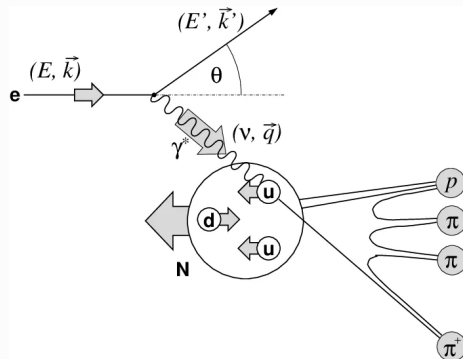
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- **Assumption (ii)** A parton must be massless to be described by a PDF. Due to this, we mainly consider the theory of high-energy scattering for massless quarks.[...]
 - Parton has to be massless to be described by a PDF.
 - Heavy quarks (quarks with large masses: charm, bottom, top) are important to correctly describe real data.
 - I consider a toy-model of the practical inclusion of the heavy quarks in this inverse problem formulation.

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Flavors of deeply inelastic scattering

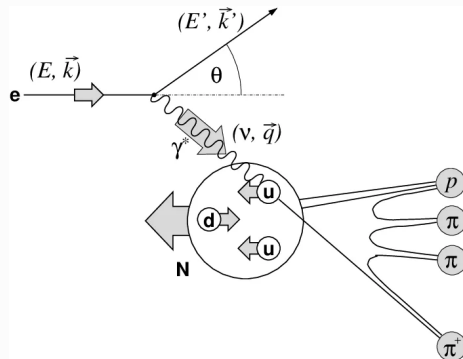
- An incoming lepton l scatters off the target proton P by exchanging a force carrier boson b :
 - The boson can be $b \in \{\gamma^*, Z, W^+, W^-\}$



Lepton–proton deeply inelastic scattering mediated by a photon γ^* (wavy line), which hits an up quark.

Flavors of deeply inelastic scattering

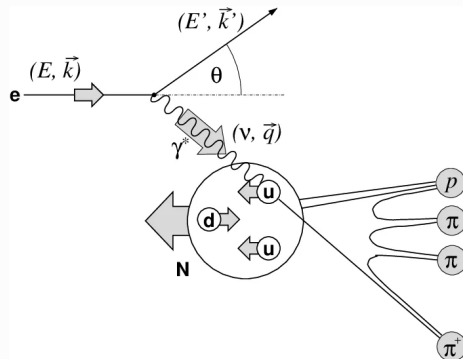
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- The QCD theory prescription of the scattering interaction is dictated by the boson b exchanged:
 - photon γ^* can see electrically charged particles. (Quarks)
 - $Z, W^\pm$ bosons can see particles with weak charge. (Quarks)



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- The proton contains gluons, how do we see those??
 - Gluons do not have electric or weak charge, so they cannot be directly probed!
 - The bosons *can* hit sea quarks, which precipitate from the gluon cloud. $\Rightarrow$ the gluon PDF has to be inferred indirectly.



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The DIS datasets that the real global analysis collaborations use

List of DIS experiment datasets that the global analyses use or consider.

- From *Eur.Phys.J. C* **82**, 428 (2022) by the NNPDF collaboration.

I am looking for a mathematical structure that unifies these within a single mathematical inverse problem.

Data set	Ref.	NNPDF3.1	NNPDF4.0	ABMP16	CT18	MSHT20
NMC F_2^d / F_2^p	[33]	✓	✓	✗	✗	✓
NMC $\sigma_{NC,p}^{NC,p}$	[34]	✓	✓	✗	✓	✓
SLAC F_2^p, F_2^d	[35, 251]	✓	✓	✗	✗	✓
BCDMS F_2^p	[36]	✓	✓	✓	✓	✓
BCDMS F_2^d	[252]	✓	✓	✗	✓	✓
BCDMS, NMC, SLAC F_L	[34, 36, 251]	✗	✗	✗	✗	✓
CHORUS $\sigma_{CC}^p, \sigma_{CC}^d$	[37]	✓	✓	✗	✗	✓
CHORUS	[253]	✗	✗	✓	✗	✗
NuTeV F_2, F_3	[254]	✗	✗	✗	✗	✓
NuTeV/CCFR $\sigma_{CC}^p, \sigma_{CC}^d$	[38]	✓	✓	✓	✓	✓
EMC F_2^p	[44]	(✓)	(✓)	✗	✗	✗
NOMAD	[111]	✗	(✓)	✓	✗	✗
CCFR $x F_3^p$	[255]	✗	✗	✗	✓	✗
CCFR F_2^p	[256]	✗	✗	✗	✓	✗
CDSHW $F_2^p, x F_3^p$	[257]	✗	✗	✗	✓	✗
E665 F_2^p, F_2^d	[258]	✗	✗	✗	✗	✓
HERA NC, CC	[259]	✗	✗	✗	✗	✓
HERA I+II $\sigma_{NC,CC}^p$	[40]	✓	✓	✓	✓	✓
HERA I+II σ_{CC}^{pd}	[145]	✗	✓	✗	(✓)	✓
HERA I+II σ_{CC}^{pd}	[145]	✗	✓	✗	(✓)	✗
HERA I+II σ_{CC}^{pd}	[41]	✓	✗	✓	✓	✗
H1 F_2^{NC}	[260]	✗	✗	✗	✓	✗
H1 $F_2^{b\bar{b}}$	[42]	✓	✗	✓	✗	✗
ZEUS $\sigma_{b\bar{b}}^{rd}$	[43]	✓	✗	✓	✗	✗
H1 F_L	[261]	✗	✗	✗	✓	✓
H1 and ZEUS F_L	[262, 263]	✗	✗	✗	✗	✓
ZEUS s20 (HQ) (1j)	[112]	✗	(✓)	✗	✗	✗
ZEUS 920 (HQ) (1j)	[113]	✗	(✓)	✗	✗	✗
H1 (LQ) (1j-2j)	[115]	✗	(✓)	✗	✗	✗
H1 (HQ) (1j-2j)	[116]	✗	(✓)	✗	✗	✗
ZEUS 920 (HQ) (2j)	[114]	✗	(✓)	✗	✗	✗

Table B.1. The fixed-target and collider DIS measurements used for PDF determination. For each PDF set, a blue tick indicates that the given dataset is included and a red cross that it is not included. A parenthesized tick denotes that a dataset was investigated but not included in the baseline fit.

Linear observables considered in this work

Process	Parton	Observable	Experiment
$e^\pm p \rightarrow e^\pm X$	$q, \bar{q}, g$	$\sigma_{r,\text{NC}}^\pm (\approx \sigma_r^\gamma), F_L^{\text{NC}}, xF_3^{\text{NC}}$	HERA I+II, ZEUS
$e^+ p \rightarrow \bar{\nu} X$	$q, \bar{q}$	$\sigma_{r,\text{CC}}^\pm$	HERA I+II
$e^\pm p \rightarrow e^\pm c\bar{c}X, e^\pm b\bar{b}X$	c, b, g	$\sigma_r^{c\bar{c}}, \sigma_r^{b\bar{b}}$	HERA I+II
$\nu(\bar{\nu})N \rightarrow \mu^-(\mu^+)X$	$q, \bar{q}$	$\sigma^{(\nu,\bar{\nu})N}, F_2^{(\nu,\bar{\nu})N}, xF_3^{(\nu,\bar{\nu})N}$	NuTeV, CHORUS
$ep \rightarrow eX, ed \rightarrow eX$	$q, \bar{q}, g$	F_L^{NC}	JLab
$\mu p \rightarrow X, \mu d \rightarrow X$	$q, \bar{q}$	Gottfried s.r. S_G, F_L^{NC}	NMC
$\nu(\bar{\nu})N \rightarrow \mu^-(\mu^+)X$	$q, \bar{q}$	GLS s.r. S_{GLS}	CCFR/NuTeV
$\nu(\bar{\nu})D \rightarrow \mu^-(\mu^+)X$	$q, \bar{q}$	Adler s.r. S_A	WA25 (BEBC)

Table: Particle scattering measurements considered in this work.

- All depend linearly on the PDFs, which we will write out explicitly to see the structure.

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A prototype coupled integral equation problem

Consider the inference of two unknown functions $f(x)$ and $g(x)$ from two distinct measurements $b_1(q)$ and $b_2(q)$ that are only sensitive to the sum and difference of the functions $f(x)$ and $g(x)$:

$$b_1(q) := \int_0^1 K_1(x, q)(f(x) + g(x))dx,$$
$$b_2(q) := \int_0^1 K_2(y, q)(f(y) - g(y))dy.$$

We formalize the integrals as linear integral operators $\mathcal{K}_i$:

$$b_i(q) = \int_0^1 K_i(x, q)(f(x) \pm g(x))dx =: \mathcal{K}_i(f \pm g) = \mathcal{K}_i(f) \pm \mathcal{K}_i(g).$$

A prototype coupled integral equation problem

Stacking the unknowns f, g and the observables b_i we get a matrix valued coupled problem:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \mathcal{K}_1 & \mathcal{K}_1 \\ \mathcal{K}_2 & -\mathcal{K}_2 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}.$$

If the integral operators $\mathcal{K}_i$ have inverse operators $\mathcal{K}_i^{-1}$, the system has the explicit solution

$$\begin{aligned} f(x) &= \frac{1}{2} \left[(\mathcal{K}_1^{-1} b_1)(x) + (\mathcal{K}_2^{-1} b_2)(x) \right] \\ g(x) &= \frac{1}{2} \left[(\mathcal{K}_1^{-1} b_1)(x) - (\mathcal{K}_2^{-1} b_2)(x) \right]. \end{aligned}$$

The main point of the paper is to show that the problem of global analysis of the PDFs can be written into a coupled integral equation system of this form.

A modified prototype

Modifying the equation generating the first row:

$$b_1(x, q) = f(x, q) + g(x, q) + \int_0^1 K_1(x, q)(f(x) + g(x))dx:$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I} + \mathcal{K}_1 & \mathbf{I} + \mathcal{K}_1 \\ \mathcal{K}_2 & -\mathcal{K}_2 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix},$$

where $(\mathbf{I} + \mathcal{K}_1)f = f(x) + (\mathcal{K}_1 f)(x)$.

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Modifying the equation generating the first row:

$$b_1(x, q) = f(x, q) + g(x, q) + \int_0^1 K_1(x, q)(f(x) + g(x))dx:$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I} + \mathcal{K}_1 & \mathbf{I} + \mathcal{K}_1 \\ \mathcal{K}_2 & -\mathcal{K}_2 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix},$$

where $(\mathbf{I} + \mathcal{K}_1)f = f(x) + (\mathcal{K}_1 f)(x)$.

If we want to enforce the normalization of the solutions, we could do so:

$$\begin{bmatrix} b_1 \\ b_2 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I} + \mathcal{K}_1 & \mathbf{I} + \mathcal{K}_1 \\ \mathcal{K}_2 & -\mathcal{K}_2 \\ \mathcal{I} & \mathcal{I} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix},$$

where $\mathcal{I}[f] := \int_0^1 f(x)dx$. So the new row gives the constraint

$$1 = \mathcal{I}[f] + \mathcal{I}[g] = \int_0^1 (f(x) + g(x))dx.$$

This is the form that we will write the global analysis of PDFs into.

Factorizing the PDFs out of the hadronic tensor

We want to increase the separation of the perturbative calculation acting as the forward operator, and the non-perturbative PDFs. The problem needs to be written in the form:

$$\mathbf{data} = L(f_g, f_d, \dots),$$

where L is some linear functional operator.

Factorizing the PDFs out of the hadronic tensor

We want to increase the separation of the perturbative calculation acting as the forward operator, and the non-perturbative PDFs. The problem needs to be written in the form:

$$\mathbf{data} = L(f_g, f_d, \dots),$$

where L is some linear functional operator.

At least in pragmatic terms in pQCD and global analysis, the PDFs are defined as the non-perturbative elements of the structure functions F_i :

$$F_i(x, Q^2) := \left(\sum_a C_i^a \otimes f_a \right) (x, Q^2),$$

and the structure functions themselves originate from the hadronic tensor:

$$W_{\mu\nu} := \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu} \right) F_1(x, Q^2) + \left(p_\mu - \frac{p \cdot q}{q^2} q_\mu \right) \left(p_\nu - \frac{p \cdot q}{q^2} q_\nu \right) F_2(x, Q^2)$$

Factorizing the PDFs out of the hadronic tensor

I want to factorize the PDFs more explicitly out of the calculation of the observables, and therefore $W_{\mu\nu}$. To this end, I formalize the Wilson coefficient convolutions as integral operators:

$$(\mathcal{C}[f])(x) = \int_x^1 C\left(\frac{x}{z}\right) f(z) \frac{dz}{z},$$

where $C\left(\frac{x}{z}\right)$ is an integration kernel, and $\mathcal{C}[f]$ is a linear functional operator on f .

Stacking the PDFs into a vector (all the PDFs in flavor space $\simeq$ parton distribution tensor):

$$\mathbf{f} := \mathbf{f}^{(6+1)} \equiv [f_g \quad f_d \quad f_u \quad f_s \quad f_c \quad f_b \quad f_t \quad f_{\bar{d}} \quad f_{\bar{u}} \quad f_{\bar{s}} \quad f_{\bar{c}} \quad f_{\bar{b}} \quad f_{\bar{t}}]^T,$$

we can represent the Wilson coefficient convolutions as a row-form integral operator:

$$\mathcal{C}_i := \left[C_i^g \quad C_i^d \quad \cdots \quad C_i^t \quad C_i^{\bar{d}} \quad \cdots \quad C_i^{\bar{t}} \right],$$

Factorizing the PDFs out of the hadronic tensor

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$$\mathcal{C}_i := \left[C_i^g \quad C_i^d \quad \dots \quad C_i^t \quad C_i^{\bar{d}} \quad \dots \quad C_i^{\bar{t}} \right],$$

which by-construction is equivalent with the canonical definition:

$$(\mathcal{C}_i \mathbf{f})(x, Q^2) = \left(\sum_a C_i^a \otimes f_a \right) (x, Q^2) \equiv F_i(x, Q^2).$$

$$W_{\mu\nu}(x, Q^2) = \left\{ \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu} \right) \mathcal{C}_1 + \left(p_\mu - \frac{p \cdot q}{q^2} q_\mu \right) \left(p_\nu - \frac{p \cdot q}{q^2} q_\nu \right) \mathcal{C}_2 \right\} \mathbf{f}.$$

Types of linear integral equations / operators that appear

The full problem exhibits diverse versions of the following kinds of linear or integral equations:

$$O_{1\text{st}}^{\sigma}(x, Q^2, y) = \int_x^1 L(z, x, y, Q^2) \mathbf{f}(z, Q^2) dz,$$

$$O_{2\text{nd}}^{\sigma}(x, Q^2, y) = \lambda(x, Q^2, y) \mathbf{f}(x, Q^2) + \int_x^1 L(z, x, y, Q^2) \mathbf{f}(z, Q^2) dz,$$

$$O_{1\text{st}}^F(x, Q^2) = \int_x^1 L(z, x, Q^2) \mathbf{f}(z, Q^2) dz,$$

$$O_{2\text{nd}}^F(x, Q^2) = \lambda(x, Q^2) \mathbf{f}(x, Q^2) + \int_x^1 L(z, x, Q^2) \mathbf{f}(z, Q^2) dz,$$

These different kinds are of some relevance when we construct the practical discretization. I need to rewrite established DIS (linear) pQCD theory prescriptions of real observables in this integral operator form **as economically as possible**.

- No new physics; Only mathematical structure of the inverse problem defined by pQCD and global analysis.

Recognizing the universal shape: Electromagnetic DIS at NLO

The canonical pQCD result for the electromagnetic DIS cross section at NLO accuracy is²:

$$\begin{aligned}\sigma_r^\gamma(x, Q^2, y) = & \sum_a e_a^2 x f_a(x, Q^2) \\ & + \sum_a e_a^2 \int_x^1 x f_a\left(\frac{x}{z}, Q^2\right) \frac{\alpha_s}{4\pi} \left(c_{2,q}^{(1)}(z)_+ - \frac{y^2}{Y_+} c_{L,q}^{(1)}(z) \right) \frac{dz}{z} \\ & + \left(\sum_a e_a^2 \right) \int_x^1 x f_g\left(\frac{x}{z}, Q^2\right) \frac{\alpha_s}{4\pi} \left(c_{2,g}^{(1)}(z)_+ - \frac{y^2}{Y_+} c_{L,g}^{(1)}(z) \right) \frac{dz}{z},\end{aligned}$$

where f_a are the PDFs to be solved, and the kernels c_i are the Wilson coefficient functions.

- This is the canonical result after some reorganization: we want the problem written in the form: $data = LinearOperator(PDFs)$.

²G. Altarelli, R. K. Ellis and G. Martinelli, Nucl. Phys. B 143 (1978) 521.

Recognizing the universal shape: Electromagnetic DIS at NLO

Writing out the sum over parton flavors:

$$\begin{aligned}\sigma_r^\gamma(x, Q^2, y) \equiv & e_d^2 x f_d(x, Q^2) + e_u^2 x f_u(x, Q^2) + e_s^2 x f_s(x, Q^2) + e_c^2 x f_c(x, Q^2) + e_b^2 x f_b(x, Q^2) + e_t^2 x f_t(x, Q^2) \\ & + e_{\bar{d}}^2 x f_{\bar{d}}(x, Q^2) + e_{\bar{u}}^2 x f_{\bar{u}}(x, Q^2) + e_{\bar{s}}^2 x f_{\bar{s}}(x, Q^2) + e_{\bar{c}}^2 x f_{\bar{c}}(x, Q^2) + e_{\bar{b}}^2 x f_{\bar{b}}(x, Q^2) + e_{\bar{t}}^2 x f_{\bar{t}}(x, Q^2) \\ & + \int_x^1 \left[e_d^2 x f_d\left(\frac{x}{z}, Q^2\right) + e_u^2 x f_u\left(\frac{x}{z}, Q^2\right) + e_s^2 x f_s\left(\frac{x}{z}, Q^2\right) + \dots \right] \frac{\alpha_s}{4\pi} \left(c_{2,q}^{(1)}(z)_+ - \frac{y^2}{Y_+} c_{L,q}^{(1)}(z) \right) \frac{dz}{z} \\ & + \left(\sum_a e_a^2 \right) \int_x^1 x f_g\left(\frac{x}{z}, Q^2\right) \frac{\alpha_s}{4\pi} \left(c_{2,g}^{(1)}(z)_+ - \frac{y^2}{Y_+} c_{L,g}^{(1)}(z) \right) \frac{dz}{z}.\end{aligned}$$

- Sum of integral equation problems that are collectively constrained by the measurement.
- From this one observable it is not possible to solve or isolate any of the unknown functions $f(x, Q^2)$. $\implies$ We need to formalize the full approach of global analysis.

Writing out the canonical physics result in the new form

We replace the sum over flavors with an inner product between integral operators and the PDFs:

$$\begin{aligned}\sigma_{r,\text{NLO}}^\gamma(x, Q^2, y) &= [\underline{e}^2 \mathcal{I}_{2-L,g} \quad e_d^2 \mathcal{I}_{2-L,q} \quad \cdots \quad e_t^2 \mathcal{I}_{2-L,q}] \begin{bmatrix} f_g(\cdot, Q^2) \\ f_d(\cdot, Q^2) \\ \vdots \\ f_t(\cdot, Q^2) \end{bmatrix} \\ &=: (\mathbf{A}_{r,\text{NLO}}^\gamma \mathbf{f})(x, Q^2, y),\end{aligned}$$

where we defined:

$$\begin{aligned}\mathcal{I}_{2-L,q}[f](x, Q^2, y) &:= x \text{Id} \circ f + \int_x^1 x f(x, Q^2) \frac{\alpha_s}{4\pi} \left(c_{2,q}^{(1)}\left(\frac{x}{z}\right)_+ - \frac{y^2}{Y_+} c_{L,q}^{(1)}\left(\frac{x}{z}\right) \right) \frac{dz}{z} \\ \mathcal{I}_{2-L,g}[f](x, Q^2, y) &:= \int_x^1 x f(x, Q^2) \frac{\alpha_s}{4\pi} \left(c_{2,g}^{(1)}\left(\frac{x}{z}\right)_+ - \frac{y^2}{Y_+} c_{L,g}^{(1)}\left(\frac{x}{z}\right) \right) \frac{dz}{z}.\end{aligned}$$

This by-construction reproduces the canonical NLO accuracy result for EM DIS. We are after the mathematical shape of the inverse problem: new form **must** reproduce literature results.

Identifying the shape in other observables at NLO

- That was electromagnetic DIS at NLO accuracy

Identifying the shape in other observables at NLO

- That was electromagnetic DIS at NLO accuracy
 - We still need:
 - Neutral current DIS at NLO (γ and Z boson exchange)
 - Charged current DIS at NLO ($W^\pm$ boson exchange)
 - Heavy quark production in DIS at NLO (charm / bottom can be tagged in experiment)
 - Neutrino–nucleus DIS at NLO ($W^\pm$ boson exchange, but with a neutrino beam)
 - Total momentum conservation sum rule
 - Flavor number conservation sum rule
- before we can move on to formalize global analysis as a mathematical inverse problem.

Neutral current DIS

Neutral current is the more general picture of the pure EM interaction: $|\gamma + Z|^2 \sim \gamma^2 + \gamma Z + Z^2$.
The neutral current reduced cross section in $e^\pm + p \rightarrow e^\pm + X$ scattering is³:

$$\sigma_{r,\text{NC}}^\pm(x, Q^2, y) = F_2^{\pm,\text{NC}}(x, Q^2) \mp \frac{Y_-}{Y_+} x F_3^{\pm,\text{NC}}(x, Q^2) - \frac{y^2}{Y_+} F_L^{\pm,\text{NC}}(x, Q^2),$$

where $F_i(x, Q^2) := (\sum_a C_i^a \otimes f_a)(x, Q^2)$ need at NLO accuracy the coefficients⁴:

$$c_{2,q}^{(1)}(z) = C_F \left\{ \frac{9}{2} + \frac{5}{2}z - 2 \left(\frac{3}{4} + \ln(z) - \ln(1-z) \right) \left(\frac{2}{1-z} - 1 - z \right) - (9 + 4\zeta(2))\delta(1-z) \right\}_+,$$

$$c_{3,q}^{(1)}(z) = c_{2,q}^{(1)}(z) - 2C_F(1+z),$$

$$c_{2,g}^{(1)}(z) = 6 - 2(4 + \ln(z) - \ln(1-z))(1 - 2z + 2z^2),$$

$$c_{L,q}^{(1)}(z) = 4C_F z,$$

$$c_{L,g}^{(1)}(z) = 8z(1-z).$$

³H1, ZEUS collaboration, Abramowicz, H., Abt, I., Adamczyk, L. *et al.* Eur. Phys. J. C 75, 580 (2015).

⁴S. Moch and J. A. M. Vermaseren, *Deep inelastic structure functions at two loops*, Nucl.Phys.B 573 (2000) 853

Neutral current forward operator at NLO

$$\sigma_{r,\text{NC}}^{\pm}(x, Q^2, y) = \left[\phi_g^{\pm} \left(\mathcal{C}_{2,g} - \frac{y^2}{Y_+} \mathcal{C}_{L,g} \right) \quad \mathcal{C}_{q,d}^{\text{NC}} \quad \cdots \quad \mathcal{C}_{q,\bar{t}}^{\text{NC}} \right] \begin{bmatrix} f_g(\cdot, Q^2) \\ f_d(\cdot, Q^2) \\ \vdots \\ f_{\bar{t}}(\cdot, Q^2) \end{bmatrix}$$

$$=: \left(\mathbf{A}_{r,\text{NC}}^{\pm} \mathbf{f} \right)(x, Q^2, y),$$

where we defined the combined integral operator for the quarks:

$$\mathcal{C}_{q,a}^{\text{NC}}[f] := \phi_{2,a}^{\pm} \left(\mathcal{C}_{2,q}[f] - \frac{y^2}{Y_+} \mathcal{C}_{L,q}[f] \right) \mp \frac{xY_-}{Y_+} \phi_{3,a}^{\pm} (-1)^{\delta_{a\bar{a}}} \mathcal{C}_{3,q}[f],$$

where the helicity-averaged effective squared parity-preserving electro-weak charge $\phi_{2,a}^{\pm}$ is:

$$\phi_{2,a}^{\pm} = e_a^2 - 2e^a g_V^a (g_V^{e,\pm}) \eta_{\gamma Z}(Q^2) + \left((g_V^a)^2 + (g_A^a)^2 \right) \left((g_V^{e,\pm})^2 + (g_A^{e,\pm})^2 \right) \eta_Z(Q^2),$$

and the effective parity-breaking counterpart of Eq. $\phi_{3,a}^{\pm}$:

$$\phi_{3,a}^{\pm} = -2e^a g_A^a g_A^{e,\pm} \eta_{\gamma Z}(Q^2) + 2g_V^a g_A^a (2g_V^{e,\pm} g_A^{e,\pm}) \eta_Z(Q^2).$$

Charged current DIS

Charged current DIS is the process $e^-(e^+) + p \rightarrow \nu(\bar{\nu}) + X$.

- Electron / positron traveling almost at the speed of light interacts with the proton and **vanishes!**
- Mediated by W^+ , W^- bosons. Scattered probe that is not seen can be tagged in experiment.

The experimentally measured reduced charged current cross section is⁵

$$\sigma_{r,CC}^{\pm}(x, Q^2, y) = \frac{Y_+}{2} F_2^{W^{\pm}}(x, Q^2) \mp \frac{Y_-}{2} x F_3^{W^{\pm}}(x, Q^2) - \frac{y^2}{2} F_L^{W^{\pm}}(x, Q^2).$$

The charged current structure functions $F_i^{W^{\pm}}$ encode the physics that

- W^+ only interacts with “down” and “anti-up” flavors ($d, s, b, \bar{u}, \bar{c}, \bar{t}$)
- W^- interacts with their counterparts ($u, c, t, \bar{d}, \bar{s}, \bar{b}$)

∴ CC gives new separation power to distinguish between flavors! (Important part of global analysis, we want to incorporate in this inverse problems formalism.)

⁵H1, ZEUS collaboration, Abramowicz, H., Abt, I., Adamczyk, L. *et al.* Eur. Phys. J. C 75, 580 (2015).

Charged current DIS forward operators at NLO

In the forward operator form the charged current reduced cross section becomes at NLO:

$$\sigma_{r,\text{CC}}^{\pm,\text{NLO}}(x, Q^2, y) = \left(\mathbf{A}_{r,\text{CC}}^{\pm,\text{NLO}} \mathbf{f} \right)(x, Q^2, y),$$

where

$$\mathbf{A}_{r,\text{CC}}^{\pm} = \begin{bmatrix} \frac{n_f Y_+}{2} \left(\mathcal{C}_{2,g} - \frac{y^2}{Y_+} \mathcal{C}_{L,g} \right) & \mathcal{C}_{q,d}^{W^+} & 0 & \mathcal{C}_{q,s}^{W^+} & 0 & \mathcal{C}_{q,b}^{W^+} & 0 & 0 & \mathcal{C}_{q,\bar{u}}^{W^+} & 0 & \mathcal{C}_{q,\bar{c}}^{W^+} & 0 & \mathcal{C}_{q,\bar{t}}^{W^+} \\ \frac{n_f Y_+}{2} \left(\mathcal{C}_{2,g} - \frac{y^2}{Y_+} \mathcal{C}_{L,g} \right) & 0 & \mathcal{C}_{q,u}^{W^-} & 0 & \mathcal{C}_{q,c}^{W^-} & 0 & \mathcal{C}_{q,t}^{W^-} & \mathcal{C}_{q,\bar{d}}^{W^-} & 0 & \mathcal{C}_{q,\bar{s}}^{W^-} & 0 & \mathcal{C}_{q,\bar{b}}^{W^-} & 0 \end{bmatrix}$$

where we defined the integral operators:

$$\begin{aligned} \mathcal{C}_{q,a}^{W^\pm}[f] &:= \frac{Y_+}{2} \mathcal{C}_{2,q}[f] \mp (-1)^{\delta_{a\bar{a}}} \frac{Y_-}{2} x \mathcal{C}_{3,q}[f] - \frac{y^2}{2} \mathcal{C}_{L,q}[f] \\ &\equiv \int_x^1 \left(\frac{Y_+}{2} c_{2,q}\left(\frac{x}{z}\right) \mp (-1)^{\delta_{a\bar{a}}} \frac{Y_-}{2} x c_{3,q}\left(\frac{x}{z}\right) - \frac{y^2}{2} c_{L,q}\left(\frac{x}{z}\right) \right) f(z) \frac{dz}{z}. \end{aligned}$$

- Flavor separation becomes clearly visible.
- The “zeros” receive corrections at NNLO, α_s^2 .

Heavy flavor production in DIS

Charm and bottom quarks are so heavy that their production can be tagged in experiment⁶:

$$e^{\pm} + p \rightarrow e^{\pm} + c\bar{c} (b\bar{b}) + X.$$

Their contributions to the structure functions are of the form

$$\begin{aligned} F_i^{Q\bar{Q}}(x, n_f+1, Q^2, m^2) = & \sum_{k=1}^{n_f} e_k^2 L_{i,q}^{\text{NS}} \left(x, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes [f_k(x, \mu^2, n_f) + f_{\bar{k}}(x, \mu^2, n_f)] \\ & + e_Q^2 \left[H_{i,q}^{\text{PS}} \left(x, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes \Sigma(x, \mu^2, n_f) \right. \\ & \left. + H_{i,g}^{\text{S}} \left(x, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes f_g(x, \mu^2, n_f) \right], \quad i \in \{2, L\}. \end{aligned}$$

⁶H1, ZEUS collaboration, H. Abramowicz *et al.* Eur. Phys. J. C 78, 473 (2018).

Charm production forward operator with heavy quark effects

Rewriting the charm production structure functions in the operator form:

$$F_i^{c\bar{c}}(x, n_f + 1 = 4, Q^2, m_c^2) = (\mathbf{B}_{i,\text{NLO}}^{c\bar{c}} \mathbf{f})(x, Q^2, m_c)$$
$$:= \begin{bmatrix} \mathcal{H}_{i,g,c}^S & \mathcal{I}_{i,d,c}^{L+H} & \mathcal{I}_{i,u,c}^{L+H} & \mathcal{I}_{i,s,c}^{L+H} & 0 & 0 & 0 & \mathcal{I}_{i,\bar{d},c}^{L+H} & \mathcal{I}_{i,\bar{u},c}^{L+H} & \mathcal{I}_{i,\bar{s},c}^{L+H} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_g(\cdot, Q^2) \\ f_d(\cdot, Q^2) \\ \vdots \\ f_{\bar{t}}(\cdot, Q^2) \end{bmatrix}$$

where

$$\mathcal{H}_{i,g,Q}^S[f](x, Q^2, m^2) := e_Q^2 \int_{ax}^1 H_{i,g}^S \left(\frac{x}{z}, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) f(z, Q^2) \frac{dz}{z},$$
$$\mathcal{I}_{i,q,Q}^{L+H}[f](x, Q^2, m^2) := \int_{ax}^1 \left\{ e_q^2 L_{i,q}^{\text{NS}} \left(\frac{x}{z}, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) + e_Q^2 H_{i,q}^{\text{PS}} \left(\frac{x}{z}, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \right\} f(z, Q^2) \frac{dz}{z}.$$

- Above assumes $n_f = 3$ light flavors, i.e. this is written in a fixed flavor number scheme (FFNS).
- I used the same form for bottom, which makes the implicit FNS to be FFNS3cb.
- My primary concern is that any FNS should be possible to implement in this formalism.

Neutrino DIS at NLO

- Neutrinos ν are notoriously difficult to detect $\implies$ Need big dense detectors and huge flux
- Detectors made of lead, iron, argon, carbon, water *are not made of free protons*
- Need to generalize the formalism for nuclear target, this is composed of Z protons and $A - Z$ neutrons. $\implies$ iso-scalar mixing.

Up and down quark PDFs are mixed as:

$$\begin{bmatrix} C'_d \\ C'_u \end{bmatrix} = \frac{1}{A} \begin{bmatrix} Z & A - Z \\ A - Z & Z \end{bmatrix} \begin{bmatrix} C_d \\ C_u \end{bmatrix} := \mathbf{w}_{A,Z} \begin{bmatrix} C_d \\ C_u \end{bmatrix},$$

which affects the full flavor basis as:

$$\mathbf{C}' = \begin{bmatrix} 1 & & & \\ & \mathbf{w}_{A,Z} & & \\ & & \mathbf{I}^{4 \times 4} & \\ & & & \mathbf{w}_{A,Z} \\ & & & & \mathbf{I}^{4 \times 4} \end{bmatrix} \mathbf{C} =: \mathbf{W}_{A,Z} \mathbf{C}.$$

Neutrino DIS forward operator at NLO

The forward operator for a nuclear target becomes:

$$\mathbf{W}_{A,Z} \mathbf{A}_{r,\text{NLO}}^{\nu(\bar{\nu})p} \\ = \begin{bmatrix} C_g^{\nu(\bar{\nu})p} & C_d^{\nu(\bar{\nu})A_N} & C_u^{\nu(\bar{\nu})A_N} & C_s^{\nu(\bar{\nu})p} & \dots & C_t^{\nu(\bar{\nu})p} & C_{\bar{d}}^{\nu(\bar{\nu})A_N} & C_{\bar{u}}^{\nu(\bar{\nu})A_N} & C_{\bar{s}}^{\nu(\bar{\nu})p} & \dots & C_{\bar{t}}^{\nu(\bar{\nu})p} \end{bmatrix}$$

where we use the short-hands:

$$C_g^{\nu(\bar{\nu})p}[f] := \frac{Y_+}{2} \left(C_{2,g}[f] - \frac{y^2}{Y_+} C_{L,g}[f] \right), \\ C_{d(\bar{d})}^{\nu(\bar{\nu})A_N}[f] := \frac{Z}{A} C_{q,d(\bar{d})}^{\nu(\bar{\nu})p}[f] + \frac{A-Z}{A} C_{q,u(\bar{u})}^{\nu(\bar{\nu})p}[f], \\ C_{u(\bar{u})}^{\nu(\bar{\nu})A_N}[f] := \frac{A-Z}{A} C_{q,d(\bar{d})}^{\nu(\bar{\nu})p}[f] + \frac{Z}{A} C_{q,u(\bar{u})}^{\nu(\bar{\nu})p}[f], \\ C_q^{\nu(\bar{\nu})p}[f] := C_{q,a}^{\nu(\bar{\nu})p}[f].$$

Identifying the shape in QCD sum rules

- Total momentum fraction conservation sum rule:

$$\int_0^1 \sum_{a \in \{g, d, u, \dots\}} x f_a(x, Q^2) dx = \int_0^1 x [f_g(x, Q^2) + f_d(x, Q^2) + f_u(x, Q^2) + f_s(x, Q^2) + f_c(x, Q^2) + f_b(x, Q^2) + f_t(x, Q^2) + f_{\bar{d}}(x, Q^2) + f_{\bar{u}}(x, Q^2) + f_{\bar{s}}(x, Q^2) + f_{\bar{c}}(x, Q^2) + f_{\bar{b}}(x, Q^2) + f_{\bar{t}}(x, Q^2)] dx \equiv 1.$$

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Define the integral operator:

$$\mathcal{I}^{(x)}[f] := \int_0^1 x f(x) dx,$$

with which the total momentum conservation sum rule becomes:

$$\mathcal{I}_{\text{t.m.c.}}[\mathbf{f}] := \begin{bmatrix} \mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} & \mathcal{I}^{(x)} \end{bmatrix} \mathbf{f}.$$

- Note: this assumes no cancellations between PDFs! Allows to change the order of sum and integral.

Flavor number sum rules

- Flavor numbers are conserved: proton has a net number of 2 up quarks, and 1 down quark.

$$\int_0^1 f_u(x, Q^2) - f_{\bar{u}}(x, Q^2) dx = N_u \equiv 2 \quad \forall Q^2 > 0.$$

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$$\int_0^1 f_u(x, Q^2) - f_{\bar{u}}(x, Q^2) dx = N_u \equiv 2 \quad \forall Q^2 > 0.$$

Define

$$\mathcal{I}^{(1)}[f](Q^2) := \int_0^1 f(x, Q^2) dx,$$

the above can be written for all flavors as a constraint given by a matrix valued integral operation:

$$\mathcal{I}_{\text{f.n.c.}}[\mathbf{f}] := \begin{bmatrix} 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} \end{bmatrix} \mathbf{f}$$

Flavor number sum rules 2

So, with:

$$\mathcal{I}_{\text{f.n.c.}}[\mathbf{f}] := \begin{bmatrix} 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} \end{bmatrix} \mathbf{f}$$

the flavor numbers of the proton can be constrained as

$$S_{\text{f.n.c.}}(Q^2) = (\mathcal{I}_{\text{f.n.c.}}\mathbf{f})(Q^2) = \int_0^1 \begin{bmatrix} f_d(x, Q^2) - f_{\bar{d}}(x, Q^2) \\ f_u(x, Q^2) - f_{\bar{u}}(x, Q^2) \\ f_s(x, Q^2) - f_{\bar{s}}(x, Q^2) \\ f_c(x, Q^2) - f_{\bar{c}}(x, Q^2) \\ f_b(x, Q^2) - f_{\bar{b}}(x, Q^2) \\ f_t(x, Q^2) - f_{\bar{t}}(x, Q^2) \end{bmatrix} dx \equiv \begin{bmatrix} N_d \\ N_u \\ N_s \\ N_c \\ N_b \\ N_t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \forall Q^2.$$

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Formulating the inverse problem of global analysis

- We have written many specific observables into linear forward operators.
- The remaining step is to formalize the inverse problem of global analysis.

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Formulating the inverse problem of global analysis

- We have written many specific observables into linear forward operators.
- The remaining step is to formalize the inverse problem of global analysis.
- We want to state a mathematical problem where the individually intractable inverse problems must be solved simultaneously to gain mathematical power to resolve the PDFs.
- This is naturally expressed as a coupled system of inverse problems.

Collecting the forward operators into a unified problem

Global analysis of the PDFs is expressed as the coupled integral equation:

$$\mathbf{O}^{\text{WD}} := \begin{bmatrix} \mathbf{O}^\sigma \\ \mathbf{O}^F \\ \mathbf{O}^S \end{bmatrix} = \int \mathbf{G}_{\text{NLO}}^{lp \rightarrow l' X} \mathbf{f},$$

where for example the cross section observable block is:

$$\mathbf{O}^\sigma := \begin{bmatrix} \sigma_r^\gamma(x, Q^2, y) \\ \sigma_{r,\text{NC}}^+(x, Q^2, y) \\ \sigma_{r,\text{NC}}^-(x, Q^2, y) \\ \sigma_{r,\text{CC}}^+(x, Q^2, y) \\ \sigma_{r,\text{CC}}^-(x, Q^2, y) \\ \sigma_r^{c\bar{c}}(x, Q^2, y) \\ \sigma_r^{b\bar{b}}(x, Q^2, y) \\ \sigma_r^{\nu_{Z^N}}(x, Q^2, y) \\ \sigma_r^{\bar{\nu}_{Z^N}}(x, Q^2, y) \end{bmatrix} = \int \begin{bmatrix} \mathbf{A}_r^\gamma(x, Q^2, y) \\ \mathbf{A}_{r,\text{NC}}^+(x, Q^2, y) \\ \mathbf{A}_{r,\text{NC}}^-(x, Q^2, y) \\ \mathbf{A}_{r,\text{CC}}^+(x, Q^2, y) \\ \mathbf{A}_{r,\text{CC}}^-(x, Q^2, y) \\ \mathbf{A}_r^{c\bar{c}}(x, Q^2, y) \\ \mathbf{A}_r^{b\bar{b}}(x, Q^2, y) \\ \mathbf{A}_r^{\nu(\bar{\nu})_{Z^N}}(x, Q^2, y) \\ \mathbf{A}_r^{\nu(\bar{\nu})_{Z^N}}(x, Q^2, y) \end{bmatrix} \mathbf{f}(\cdot, Q^2).$$

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and the structure function block is:

$$\mathbf{O}^F := \begin{bmatrix} F_L^\gamma(x, Q^2) \\ xF_3^{\text{NC}}(x, Q^2) \\ F_2^{\nu N}(x, Q^2) \\ xF_3^{\nu N}(x, Q^2) \\ F_2^{\bar{\nu} N}(x, Q^2) \\ xF_3^{\bar{\nu} N}(x, Q^2) \end{bmatrix} = \int \begin{bmatrix} \mathbf{B}_L^\gamma(x, Q^2) \\ \mathbf{B}_{3,\text{NC}}^\pm(x, Q^2) \\ \mathbf{B}_2^{\nu Z^A N}(x) \\ \mathbf{B}_3^{\nu Z^A N}(x) \\ \mathbf{B}_2^{\bar{\nu} Z^A N}(x) \\ \mathbf{B}_3^{\bar{\nu} Z^A N}(x) \end{bmatrix} \mathbf{f}(\cdot, Q^2)$$

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and the sum rule constraint block is:

$$\mathbf{O}^S := \begin{bmatrix} S_{\text{t.m.c.}}(Q^2) \\ S_{\text{f.n.c.}}(Q^2) \\ S_{\text{GLS}}(Q^2) \\ S_A(Q^2) \\ S_G(Q^2) \end{bmatrix} = \int \begin{bmatrix} \mathcal{I}_{\text{t.m.c.}} \\ \mathcal{I}_{\text{f.n.c.}} \\ \mathcal{I}_{\text{GLS}} \\ \mathcal{I}_A \\ \mathcal{I}_G \end{bmatrix} \mathbf{f}(\cdot, Q^2).$$

Each element in the kernel matrix is a row-form integral operator we had to rewrite from the established physics results, which by-construction reproduce the original QCD results.

The expanding to see some structure of the inverse problem

The mathematical shape of global analysis is a huge coupled system of integral equations:

$$\mathbf{O}^{\text{WD}} = \int_0^1 \mathbf{G}_{\text{NLO}}^{lp \rightarrow l'X}(z, x, Q^2, y) \mathbf{f}(z, Q^2) \frac{dz}{z}$$

$$\Leftrightarrow \begin{bmatrix} \sigma_r^\gamma(x, Q^2, y) \\ \vdots \\ F_L^\gamma(x, Q^2) \\ \vdots \\ S_{\text{t.m.c.}}(Q^2) \\ \vdots \end{bmatrix} = \int_0^1 \begin{bmatrix} e_\sim^2 \mathcal{I}_{2-L,g} \theta_x & e_d^2 \mathcal{I}_{2-L,q} \theta_x & \cdots & e_t^2 \mathcal{I}_{2-L,q} \theta_x \\ \vdots & \ddots & \cdots & \vdots \\ e_\sim^2 \mathcal{I}_{L,g} \theta_x & e_d^2 \mathcal{I}_{L,q} \theta_x & \cdots & e_t^2 \mathcal{I}_{L,q} \theta_x \\ \vdots & \ddots & \cdots & \vdots \\ z^2 & z^2 & \cdots & z^2 \\ \vdots & \ddots & \cdots & \vdots \end{bmatrix} \begin{bmatrix} f_g(z, Q^2) \\ f_d(z, Q^2) \\ \vdots \\ f_{\bar{t}}(z, Q^2) \end{bmatrix} \frac{dz}{z}.$$

This system-level integral equation structure has not been identified previously!?

- Global analysis of the PDFs has always been a large number of formulae to compute and compare against data. Here we see a substantial amount of structure, to try and solve.

DIS world data integral equation system

$$\mathbf{o}^{\text{WD},lp \rightarrow l'X}(x, Q^2, y) =$$

$$\begin{bmatrix} e^2 \mathcal{I}_{2-L,g} & e_d^2 \mathcal{I}_{2-L,q} & e_u^2 \mathcal{I}_{2-L,q} & e_s^2 \mathcal{I}_{2-L,q} & e_c^2 \mathcal{I}_{2-L,q} & e_b^2 \mathcal{I}_{2-L,q} & e_t^2 \mathcal{I}_{2-L,q} & e_d^2 \mathcal{I}_{2-L,q} & e_u^2 \mathcal{I}_{2-L,q} & e_s^2 \mathcal{I}_{2-L,q} & e_c^2 \mathcal{I}_{2-L,q} & e_b^2 \mathcal{I}_{2-L,q} & e_t^2 \mathcal{I}_{2-L,q} \\ \phi_g^+ \mathcal{C}_{2-L,g} & \mathcal{C}_{q,d}^{+,NC} & \mathcal{C}_{q,u}^{+,NC} & \mathcal{C}_{q,s}^{+,NC} & \mathcal{C}_{q,c}^{+,NC} & \mathcal{C}_{q,b}^{+,NC} & \mathcal{C}_{q,t}^{+,NC} & \mathcal{C}_{q,d}^{+,NC} & \mathcal{C}_{q,u}^{+,NC} & \mathcal{C}_{q,s}^{+,NC} & \mathcal{C}_{q,c}^{+,NC} & \mathcal{C}_{q,b}^{+,NC} & \mathcal{C}_{q,t}^{+,NC} \\ \phi_g^- \mathcal{C}_{2-L,g} & \mathcal{C}_{q,d}^{-,NC} & \mathcal{C}_{q,u}^{-,NC} & \mathcal{C}_{q,s}^{-,NC} & \mathcal{C}_{q,c}^{-,NC} & \mathcal{C}_{q,b}^{-,NC} & \mathcal{C}_{q,t}^{-,NC} & \mathcal{C}_{q,d}^{-,NC} & \mathcal{C}_{q,u}^{-,NC} & \mathcal{C}_{q,s}^{-,NC} & \mathcal{C}_{q,c}^{-,NC} & \mathcal{C}_{q,b}^{-,NC} & \mathcal{C}_{q,t}^{-,NC} \\ \frac{y_+}{2} \mathcal{C}_{2-L,g} & \mathcal{C}_{q,d}^{W^+} & 0 & \mathcal{C}_{q,s}^{W^+} & 0 & \mathcal{C}_{q,b}^{W^+} & 0 & 0 & \mathcal{C}_{q,u}^{W^+} & 0 & \mathcal{C}_{q,c}^{W^+} & 0 & \mathcal{C}_{q,t}^{W^+} \\ \frac{y_-}{2} \mathcal{C}_{2-L,g} & 0 & \mathcal{C}_{q,u}^{W^-} & 0 & \mathcal{C}_{q,c}^{W^-} & 0 & \mathcal{C}_{q,t}^{W^-} & \mathcal{C}_{q,d}^{W^-} & 0 & \mathcal{C}_{q,s}^{W^-} & 0 & \mathcal{C}_{q,b}^{W^-} & 0 \\ \mathcal{H}_{r,g,c}^S & \mathcal{I}_{r,d,c}^{L+H} & \mathcal{I}_{r,u,c}^{L+H} & \mathcal{I}_{r,s,c}^{L+H} & 0 & 0 & 0 & \mathcal{I}_{r,d,c}^{L+H} & \mathcal{I}_{r,u,c}^{L+H} & \mathcal{I}_{r,s,c}^{L+H} & 0 & 0 & 0 \\ \mathcal{H}_{r,g,b}^S & \mathcal{I}_{r,d,b}^{L+H} & \mathcal{I}_{r,u,b}^{L+H} & \mathcal{I}_{r,s,b}^{L+H} & 0 & 0 & 0 & \mathcal{I}_{r,d,b}^{L+H} & \mathcal{I}_{r,u,b}^{L+H} & \mathcal{I}_{r,s,b}^{L+H} & 0 & 0 & 0 \\ \mathcal{C}_g^{\nu p} & \mathcal{C}_d^{\nu \frac{A}{2} N} & \mathcal{C}_u^{\nu \frac{A}{2} N} & \mathcal{C}_s^{\nu p} & \mathcal{C}_c^{\nu p} & \mathcal{C}_b^{\nu p} & \mathcal{C}_t^{\nu p} & \mathcal{C}_d^{\nu \frac{A}{2} N} & \mathcal{C}_u^{\nu \frac{A}{2} N} & \mathcal{C}_s^{\nu p} & \mathcal{C}_c^{\nu p} & \mathcal{C}_b^{\nu p} & \mathcal{C}_t^{\nu p} \\ \mathcal{C}_g^{\bar{\nu} p} & \mathcal{C}_d^{\bar{\nu} \frac{A}{2} N} & \mathcal{C}_u^{\bar{\nu} \frac{A}{2} N} & \mathcal{C}_s^{\bar{\nu} p} & \mathcal{C}_c^{\bar{\nu} p} & \mathcal{C}_b^{\bar{\nu} p} & \mathcal{C}_t^{\bar{\nu} p} & \mathcal{C}_d^{\bar{\nu} \frac{A}{2} N} & \mathcal{C}_u^{\bar{\nu} \frac{A}{2} N} & \mathcal{C}_s^{\bar{\nu} p} & \mathcal{C}_c^{\bar{\nu} p} & \mathcal{C}_b^{\bar{\nu} p} & \mathcal{C}_t^{\bar{\nu} p} \\ 2 \frac{\alpha_s}{\pi} e^2 \mathcal{I}_{L,g} & e_d^2 x \mathcal{I}_{L,q} & e_u^2 x \mathcal{I}_{L,q} & e_s^2 x \mathcal{I}_{L,q} & e_c^2 x \mathcal{I}_{L,q} & e_b^2 x \mathcal{I}_{L,q} & e_t^2 x \mathcal{I}_{L,q} & e_d^2 x \mathcal{I}_{L,q} & e_u^2 x \mathcal{I}_{L,q} & e_s^2 x \mathcal{I}_{L,q} & e_c^2 x \mathcal{I}_{L,q} & e_b^2 x \mathcal{I}_{L,q} & e_t^2 x \mathcal{I}_{L,q} \\ 0 & \phi_{3,d}^{\pm} \mathcal{C}_{3,q} & \phi_{3,u}^{\pm} \mathcal{C}_{3,q} & \phi_{3,s}^{\pm} \mathcal{C}_{3,q} & \phi_{3,c}^{\pm} \mathcal{C}_{3,q} & \phi_{3,b}^{\pm} \mathcal{C}_{3,q} & \phi_{3,t}^{\pm} \mathcal{C}_{3,q} & -\phi_{3,d}^{\pm} \mathcal{C}_{3,q} & -\phi_{3,u}^{\pm} \mathcal{C}_{3,q} & -\phi_{3,s}^{\pm} \mathcal{C}_{3,q} & -\phi_{3,c}^{\pm} \mathcal{C}_{3,q} & -\phi_{3,b}^{\pm} \mathcal{C}_{3,q} & -\phi_{3,t}^{\pm} \mathcal{C}_{3,q} \\ \frac{1}{2} \mathcal{C}_{2,g} & \frac{Z}{A} \mathcal{C}_{2,q} & \frac{A-Z}{A} \mathcal{C}_{2,q} & \mathcal{C}_{2,q} & 0 & \mathcal{C}_{2,q} & 0 & \frac{A-Z}{A} \mathcal{C}_{2,q} & \frac{Z}{A} \mathcal{C}_{2,q} & 0 & \mathcal{C}_{2,q} & 0 & \mathcal{C}_{2,q} \\ 0 & \frac{Z}{A} \mathcal{C}_{3,q} & \frac{A-Z}{A} \mathcal{C}_{3,q} & \mathcal{C}_{3,q} & 0 & \mathcal{C}_{3,q} & 0 & -\frac{A-Z}{A} \mathcal{C}_{3,q} & -\frac{Z}{A} \mathcal{C}_{3,q} & 0 & -\mathcal{C}_{3,q} & 0 & -\mathcal{C}_{3,q} \\ \frac{1}{2} \mathcal{C}_{2,g} & \frac{A-Z}{A} \mathcal{C}_{2,q} & \frac{Z}{A} \mathcal{C}_{2,q} & 0 & \mathcal{C}_{2,q} & 0 & \mathcal{C}_{2,q} & \frac{Z}{A} \mathcal{C}_{2,q} & \frac{A-Z}{A} \mathcal{C}_{2,q} & \mathcal{C}_{2,q} & 0 & \mathcal{C}_{2,q} & 0 \\ 0 & \frac{A-Z}{A} \mathcal{C}_{3,q} & \frac{Z}{A} \mathcal{C}_{3,q} & 0 & \mathcal{C}_{3,q} & 0 & \mathcal{C}_{3,q} & -\frac{Z}{A} \mathcal{C}_{3,q} & -\frac{A-Z}{A} \mathcal{C}_{3,q} & -\mathcal{C}_{3,q} & 0 & -\mathcal{C}_{3,q} & 0 \\ 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} \\ \mathcal{I}(x) & \mathcal{I}(x) & \mathcal{I}(x) & \mathcal{I}(x) & \mathcal{I}(x) & \mathcal{I}(x) & \mathcal{I}(x) & \mathcal{I}(x) & \mathcal{I}(x) & \mathcal{I}(x) & \mathcal{I}(x) & \mathcal{I}(x) & \mathcal{I}(x) \\ 0 & \mathcal{I}^{(1)} & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & -\mathcal{I}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & -\mathcal{I}^{(1)} & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & \mathcal{I}^{(1)} & -\mathcal{I}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & -\mathcal{I}^{(1)} & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 & -\mathcal{I}^{(1)} & \mathcal{I}^{(1)} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_g(\cdot, Q^2) \\ f_d(\cdot, Q^2) \\ f_u(\cdot, Q^2) \\ f_s(\cdot, Q^2) \\ f_c(\cdot, Q^2) \\ f_b(\cdot, Q^2) \\ f_t(\cdot, Q^2) \\ f_{\bar{d}}(\cdot, Q^2) \\ f_{\bar{u}}(\cdot, Q^2) \\ f_{\bar{s}}(\cdot, Q^2) \\ f_{\bar{c}}(\cdot, Q^2) \\ f_{\bar{b}}(\cdot, Q^2) \\ f_{\bar{t}}(\cdot, Q^2) \end{bmatrix}$$

Number of observables and the rank of the forward operator

In this formulation we can consider the rank of the coupled system of integral equations generated by the world data forward operator $\mathbf{G}_{\text{NLO}}^{lp \rightarrow l' X}(z, x, Q^2, y)$.

- 13 unknown PDFs would require a minimum of 13 integral equation constraints in $\mathbf{G}$?

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Open questions: What are the conditions for the full coupled integral equation problem to have any solutions? Or a unique solution?

- Or does the full problem have non-trivial null space?
 - Mathematical characterization of features of PDFs not constrainable by global analysis: changes in the flavor null space will keep the world data invariant!

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Physics constraint for the solutions from QCD

Perturbative QCD predicts the Q^2 -dependence of the PDFs:

$$\frac{\partial}{\partial \log Q^2} f_i(x, Q^2) = \frac{\alpha_s(Q^2)}{4\pi} \sum_j^{n_f} \int_x^1 P_{ij}(z, Q) f_j\left(\frac{z}{x}, Q^2\right) \frac{dz}{z}, \quad (\text{DGLAP})$$

where the kernel functions P_{ij} depend on the partons i and j .

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where the kernel functions P_{ij} depend on the partons i and j . This can be (and has been) written in a matrix form (here $n_f = 6$):

$$\frac{\partial}{\partial \log Q^2} \mathbf{f}(x, Q^2) = \frac{\partial}{\partial \log Q^2} \begin{bmatrix} f_g(x, Q^2) \\ f_d(x, Q^2) \\ \vdots \\ f_{\bar{t}}(x, Q^2) \end{bmatrix} = \begin{bmatrix} \mathcal{P}_{gg} & \mathcal{P}_{gq} & \cdots & \mathcal{P}_{gq} \\ \mathcal{P}_{qg} & \mathcal{P}_{qq} & \cdots & \mathcal{P}_{qq} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{P}_{qg} & \mathcal{P}_{qq} & \cdots & \mathcal{P}_{qq} \end{bmatrix} \begin{bmatrix} f_g(\cdot, Q^2) \\ f_d(\cdot, Q^2) \\ \vdots \\ f_{\bar{t}}(\cdot, Q^2) \end{bmatrix} =: \mathbf{P}_{\text{DGLAP}} \mathbf{f}.$$

- Conventionally built-in as a hard requirement in PDF fits.
 - PDFs parametrized at scale Q_0^2 , solved with DGLAP for $Q^2 \geq Q_0^2$, and compared with data.

Evolution of internal d.o.f., and their connection to observables

- DGLAP equation for internal degrees of freedom:

$$\frac{\partial}{\partial \log Q^2} \mathbf{f}(x, Q^2) = \frac{\partial}{\partial \log Q^2} \begin{bmatrix} f_g(x, Q^2) \\ f_d(x, Q^2) \\ \vdots \\ f_{\bar{t}}(x, Q^2) \end{bmatrix} = \begin{bmatrix} \mathcal{P}_{gg} & \mathcal{P}_{gq} & \cdots & \mathcal{P}_{gq} \\ \mathcal{P}_{qg} & \mathcal{P}_{qq} & \cdots & \mathcal{P}_{qq} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{P}_{qg} & \mathcal{P}_{qq} & \cdots & \mathcal{P}_{qq} \end{bmatrix} \begin{bmatrix} f_g(\cdot, Q^2) \\ f_d(\cdot, Q^2) \\ \vdots \\ f_{\bar{t}}(\cdot, Q^2) \end{bmatrix} =: \mathbf{P}_{\text{DGLAP}} \mathbf{f}.$$

- System of integral equations for their connection to experimental observables:

$$\begin{bmatrix} \sigma_r^\gamma(x, Q^2, y) \\ \vdots \\ F_L^\gamma(x, Q^2) \\ \vdots \\ S_{\text{t.m.c.}}(Q^2) \\ \vdots \end{bmatrix} = \int_0^1 \begin{bmatrix} e_{\sim}^2 \mathcal{I}_{2-L,g} \theta_x & e_d^2 \mathcal{I}_{2-L,q} \theta_x & \cdots & e_{\bar{t}}^2 \mathcal{I}_{2-L,q} \theta_x \\ \vdots & \ddots & \cdots & \vdots \\ e_{\sim}^2 \mathcal{I}_{L,g} \theta_x & e_d^2 \mathcal{I}_{L,q} \theta_x & \cdots & e_{\bar{t}}^2 \mathcal{I}_{L,q} \theta_x \\ \vdots & \ddots & \cdots & \vdots \\ z^2 & z^2 & \cdots & z^2 \\ \vdots & \ddots & \cdots & \vdots \end{bmatrix} \begin{bmatrix} f_g(z, Q^2) \\ f_d(z, Q^2) \\ \vdots \\ f_{\bar{t}}(z, Q^2) \end{bmatrix} \frac{dz}{z}.$$

A toy model FNS to include some heavy flavor effects

In this work I consider a toy-model of a flavor number scheme (FNS) with some heavy flavor effects:

$$O^{3\text{cb}+\text{ZM}-\text{VFNS}}(Q^2) := \begin{cases} O^{\text{FFNS3cb}}(Q^2) & \text{if } Q^2 < m_b^2 \\ O^{(5)}(Q^2) & \text{if } m_b^2 \leq Q^2 < m_t^2 \\ O^{(6)}(Q^2) & \text{if } m_t^2 \leq Q^2, \end{cases}$$

where in the lowest interval heavy flavor effects should be systematically included for 3 light flavors, and then in the two higher regimes in Q^2 all quarks are taken as massless and only their active number changes.

- Observable O is defined piece-wise on intervals in Q^2 .
- Q^2 of measurement dictates which flavors contribute.

The *real* full inverse problem with a flavor number scheme

With the flavor number scheme, the integral equation system of global analysis becomes:

$$\mathbf{O}_{\text{FNS}}^{\text{WD}} := \begin{bmatrix} \mathbf{O}_{Q^2 < m_b^2}^{\text{WD}} \\ \mathbf{O}_{m_b^2 \leq Q^2 < m_t^2}^{\text{WD}} \\ \mathbf{O}_{m_t^2 \leq Q^2}^{\text{WD}} \end{bmatrix} = \int \begin{bmatrix} \mathbf{G}_{Q^2 < m_b^2}^{lp \rightarrow l' X} \\ \mathbf{G}_{m_b^2 \leq Q^2 < m_t^2}^{lp \rightarrow l' X} \\ \mathbf{G}_{m_t^2 \leq Q^2}^{lp \rightarrow l' X} \end{bmatrix} \mathbf{f} =: \int \mathbf{G}_{\text{FNS}}^{lp \rightarrow l' X} \mathbf{f}.$$

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The functional optimization problem then becomes with the DGLAP PDE constraint:

$$\begin{aligned} & \underset{\mathbf{f}, \mathbf{y}}{\text{minimize}} \left\| \mathbf{O}_{\text{FNS}}^{\text{WD}} - \int \mathbf{G}_{\text{FNS}}^{lp \rightarrow l' X} \mathbf{f} \right\|_{L^2(\Omega)}^2 + \lambda \|\mathbf{y}\|_{L^2(\Omega)}^2, \\ & \text{subject to } \mathbf{y} := \frac{\partial}{\partial \log Q^2} \mathbf{f}(x, Q^2) - \mathbf{P}_{\text{FNS}}^{\text{DGLAP}} \mathbf{f}. \end{aligned}$$

- In the paper I examined other FNSs: I hope that the approach is capable of proper general variable flavor number schemes, above is merely an illustrative toy-model of the problem.

Any observable that satisfies the collinear factorization of pQCD:

$$F_i^b(x, Q^2) = \sum_a \int_x^1 C_{i,a}^b \left(\frac{x}{z}, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) f_{a,H}(z, \mu^2) \frac{dz}{z} + O(1 \text{ GeV}/Q)$$

is amenable to this approach, and since the perturbative corrections go to C_i , this approach is at least theoretically possible also at higher orders (NNLO, N^3LO ,..).

- State of the art of PDF extraction is at N^3LO accuracy.
 - Corrections up to α_s^3 included in the perturbative expansion.
 - Any interesting alternative methodology of global analysis should be capable of higher orders, which this formalism would seem capable of.

On the solution of the parton level integral equations

Wiener–Hopf integral equations can be of first or second-kind:

$$\begin{aligned}\int_0^\infty k(x-y)f(y)dy &= g(x), \quad x \in (0, \infty), \\ f(x) - \int_0^\infty k(x-y)f(y)dy &= g(x), \quad x \in (0, \infty),\end{aligned}$$

where $k(x-y)$ is known as a difference kernel. Conditions for the existence of (unique) solutions are known in many cases⁷.

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where $k(x-y)$ is known as a difference kernel. Conditions for the existence of (unique) solutions are known in many cases⁷. The Wilson coefficient convolutions can be rewritten:

$$(C \otimes f)(x) := \int_x^1 C\left(\frac{x}{z}\right) f(z) \frac{dz}{z} \equiv - \int_0^{\ln(x)} \tilde{C}(\tilde{x} - \tilde{z}) \tilde{f}(\tilde{z}) d\tilde{z},$$

which would appear to be quite closely related to the Wiener–Hopf equations. $\tilde{C}$ can be a plus-distribution (i.e. not in L^1), so inspection of applicability of existing results needed.

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Further stacking is done for datapoints in y , and for different observables to form a discretized formulation of the full world data problem.

Stacking form

Starting from an observable of the form: $O_a(x) = \int_x^1 C\left(\frac{x}{z}\right) f_a(z) \frac{dz}{z}$

➊ Nyström quadrature method: Discretization in z and x .

- Define quadrature $\mathcal{Q}_n := \sum_{j=1}^n \omega_j \frac{f(z_j)}{z_j} \approx \int_x^1 f(z) \frac{dz}{z}$
- Convolution by quadrature $(\mathcal{C}_n f)(x) := \mathcal{Q}_n\left(C\left(\frac{x}{\cdot}\right) f\right) := \sum_{j=1}^n \omega_j \frac{C\left(\frac{x}{z_j}\right)}{z_j} f(z_j)$
- Discretize both in z and x : $M_{jk} := \omega_j \frac{C\left(\frac{x_k}{z_j}\right)}{z_j}$, $1 \leq j \leq n$, $1 \leq k \leq m$
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 $\underline{Q}_a := (O_a(x_1), \dots, O_a(x_m))^T = M \underline{f}_a$

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- Stack: $\mathbf{f}_N := [\mathbf{f}_n(Q_1^2) \quad \mathbf{f}_n(Q_2^2) \quad \dots \quad \mathbf{f}_n(Q_N^2)]^T$
- $\mathbf{A}_{a,\nu ml} := A_a(z_\nu, x_m, Q_l^2)$; forward operator is discretized in z, x, Q^2 .
- $O(x_m, Q_l^2) = \mathbf{A}_{ml} \mathbf{f}_n(Q_l^2) = \sum_a \sum_\nu A_{a,\nu ml} f_a(z_\nu, Q_l^2)$.

Stacking form 2 – discretization boogaloo

- ④ Discretization in y . (Repeat the previous steps for each node in y .)

$$\bullet O(x, Q^2, y) = \begin{bmatrix} O_{MN}(y_1) \\ \vdots \\ O_{MN}(y_K) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{MN}(y_1) \\ \vdots \\ \mathbf{A}_{MN}(y_K) \end{bmatrix} \mathbf{f}_N =: \mathbf{A}_{MNK} \mathbf{f}_N,$$

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- Repeat for any other measurements or sum rule constraints to include.

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The discretized problem becomes a matrix $\times$ vector form linear inverse problem:

$$\bullet \begin{bmatrix} \sigma_r^\gamma(x_m, Q_n^2, y_k)_{1 \leq m \leq M, 1 \leq n \leq N, 1 \leq k \leq K} \\ F_L^\gamma(x_m, Q_n^2)_{1 \leq m \leq M, 1 \leq n \leq N} \\ S_{\text{t.m.c.}}(Q_n^2)_{1 \leq n \leq N} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{r, \text{NLO}, (MNK)}^\gamma \\ \mathbf{B}_{L, (MN)}^\gamma \\ \mathbf{Y}_{n, N}^{\text{t.m.c.}} \end{bmatrix} \mathbf{f}_N$$

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A “faint similarity” in the maths of a CT scan and global analysis

Computed tomography is built on the mathematical foundation of the X-ray transform:

$$Xf(\theta, x) := \int_{-\infty}^{\infty} f(x + t\theta) dt,$$

where $f(x)$ is the 3D distribution of X-ray attenuation, and Xf is measured in the scan.

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where $f(x)$ is the 3D distribution of X-ray attenuation, and Xf is measured in the scan. In this framing the extraction of the PDFs from DIS world data looks like a huge coupled system of integral equations:

$$\begin{bmatrix} \sigma_r^\gamma(x, Q^2, y) \\ \vdots \\ F_L^\gamma(x, Q^2) \\ \vdots \\ S_{\text{t.m.c.}}(Q^2) \\ \vdots \end{bmatrix} = \int_0^1 \begin{bmatrix} e_{\sim}^2 \mathcal{I}_{2-L,g} \theta_x & e_d^2 \mathcal{I}_{2-L,q} \theta_x & \cdots & e_{\bar{t}}^2 \mathcal{I}_{2-L,q} \theta_x \\ \vdots & \ddots & \cdots & \vdots \\ e_{\sim}^2 \mathcal{I}_{L,g} \theta_x & e_d^2 \mathcal{I}_{L,q} \theta_x & \cdots & e_{\bar{t}}^2 \mathcal{I}_{L,q} \theta_x \\ \vdots & \ddots & \cdots & \vdots \\ z^2 & z^2 & \cdots & z^2 \\ \vdots & \ddots & \cdots & \vdots \end{bmatrix} \begin{bmatrix} f_g(z, Q^2) \\ f_d(z, Q^2) \\ \vdots \\ f_{\bar{t}}(z, Q^2) \end{bmatrix} \frac{dz}{z}.$$

Discussion and outlook

- I formulated the extraction of PDFs from DIS data into a mathematical inverse problem:
 - Needed to recognize the high level structure present in a diverse set of DIS observables.
 - Reviewed the DIS observables at LO and NLO accuracy, and wrote them in the new inverse problem formalism. N.B. all NLO DIS observables in unified notation and form.
 - Formalized an inverse problem that needs to:
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 - Can the forward operator be proven to have full rank with sufficiently many observable and sum rule constraints? (Might depend on the perturbative accuracy used?)
 - OR: will we see a non-trivial null space of the world data operator $\mathbf{G}$? ($\mathbf{G}\mathbf{x} = 0 \forall \mathbf{x} \in \ker \mathbf{G}$), i.e. any feature in the null space is not seen in observables.

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 - Inverse theory for the integral equations: what can we say about existence and uniqueness of solutions? At the parton level? How about at the full coupled problem level?
 - Is the Wiener–Hopf framing useful?

Discussion and outlook

- Novel possibilities in QCD physics?
 - Generalized solution of the PDFs from world data in a functional sense: mathematical path towards “indirect sensing”?
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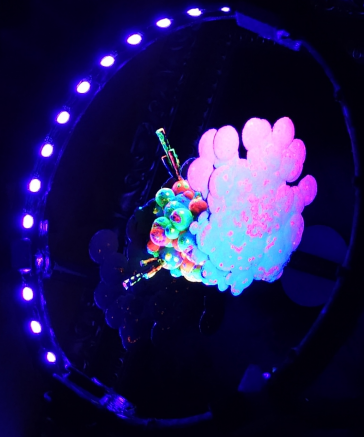
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- Is the “parton distribution tensor” physically significant?
 - Constructed purely as a mathematical convenience.
 - Quite neatly extracts the non-perturbative PDFs out of the hadronic tensor $W^{\mu\nu}$?



Thank you for listening!

Questions?
Comments?

Contact:

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[hhannine.github.io](https://github.com/hhannine)

Appendix

Plus-distributions

The Wilson coefficient functions acting as kernels can contain plus-distributions, which are defined as:

$$\int_0^1 a(z)_+ f(z) dz := \int_0^1 a(x) (f(z) - f(1)) dz.$$

Writing this out for the convolutions for the ratio of the arguments in the interval $[x, 1]$:

$$\begin{aligned}(c_+ \otimes f)(x) &:= \int_x^1 c_+(z) f\left(\frac{x}{z}\right) \frac{dz}{z} \\&= \int_x^1 C\left(\frac{x}{z}\right) \left(f(z) - \frac{x}{z} f(x)\right) \frac{dz}{z} - f(x) \int_0^x C(z) dz \\&=: \int_x^1 C\left(\frac{x}{z}\right) \int_x^1 \left(\delta(z' - z) - \frac{x}{z} \delta(z' - x)\right) f(z') dz' \frac{dz}{z} - f(x) c(x).\end{aligned}$$

These are used to cancel out divergences between terms that appear when integrating

$$\int_x^{1-\varepsilon} c_+(z) dz \xrightarrow{\varepsilon \rightarrow 0} \infty.$$