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Turning the dipole picture of deep inelastic scattering into a definition for an integral transform

Center of Excellence in Quark Matter seminar

Henri Hänninen

November 28, 2025

Based on joint work with

Antti Kykkänen (Rice U.) & Hjørdis Schlüter (Helsinki U.)

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Goal of the talk: introduce the new inverse problem approach

PHYSICAL REVIEW D **112,** 094026 (2025)

Reconstruction of the dipole amplitude in the dipole picture as a mathematical inverse problem

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We show that the inference problem of constraining the dipole amplitude with inclusive deep inelastic scattering data can be written into a discrete linear inverse problem, in an analogous manner as can be done for computed tomography. To this formulation of the problem, we apply standard inverse problems methods and algorithms to reconstruct known dipole amplitudes from simulated reduced cross section data with realistic precision. The main difference of this approach to previous works is that this implementation does not require any fit parametrization of the dipole amplitude. The freedom from parametrization also enables us for the first time to quantify the uncertainties of the inferred dipole amplitude in a novel more general framework. This mathematical approach to small-x phenomenology opens a path to parametrization bias-free inference of the dipole amplitude from HERA and Electron-Ion Collider data.

DOI: 10.1103/7fhf-7fp4

A motivating prelude: diffractive meson electroproduction

S. Munier, A.M. Stasto, A.H. Mueller wanted to extract the dipole amplitude from vector meson production data without an assumed model parametrization [1]:

$$\int \frac{\mathrm{d}^2\Delta}{(2\pi)^2}\sqrt{\frac{\mathrm{d}\sigma}{\mathrm{d}t}}e^{-i\Delta\mathbf{b}} = \frac{1}{\sqrt{16\pi}}\sum_{h,\bar{h}}\int \mathrm{d}^2\mathbf{r}\mathrm{d}r\psi_{\gamma^*}^{h,\bar{h}}(z,r,Q)2[1-S(x,r,b)]\psi_V^{h,\bar{h}}(z,\mathbf{r}),$$

[1] Nuclear Physics B Volume 603, Issues 1–2, 4 June 2001, Pages 427-445; https://doi.org/10.1016/S0550-3213(01)00168-7

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which, after defining a probability measure p using the light-cone wavefunctions $\psi_{\gamma^*,V}$ and its normalization $N(Q)$, they invert into

$$\langle S(x,r,b)\rangle_p = 1 - \frac{1}{2N(Q)\pi^{3/2}}\int \mathrm{d}^2\Delta e^{-i\Delta\mathbf{b}}\sqrt{\frac{\mathrm{d}\sigma}{\mathrm{d}t}}.$$

This enabled the extraction of information about the r-averaged dipole amplitude from HERA data. Key step here is the invertibility of the Fourier transform.

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"Can one do something if the integral relation is not (known to be) analytically invertible?"

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Nomenclature: forward vs inverse problem

We will consider a linear equation:

$$\mathbf{b} = \mathbf{A}\mathbf{x},$$

where $\mathbf{b}$ is data acquired from an experiment, $\mathbf{A}$ is a so-called forward operator, here matrix valued, and $\mathbf{x}$ is the unknown we want to solve from the data (to indirectly measure).

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- Forward problem $\hat{=}$ physical theory or model that enables calculation of predictions of data if the quantity $\mathbf{x}$ is known.
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We will write the leading order dipole picture of DIS (*as you know it!*) in this form, where the data $\mathbf{b}$ is the reduced cross section $\sigma_r(Q, x)$, and the unknown $\mathbf{x}$ is the dipole amplitude $N(r, x)$.

Linear inverse problem? Analogous with computed tomography?

What do you mean computed tomography has something in common with deep inelastic scattering in the dipole picture??

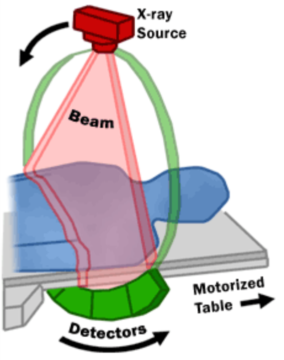
Linear inverse problem? Analogous with computed tomography?

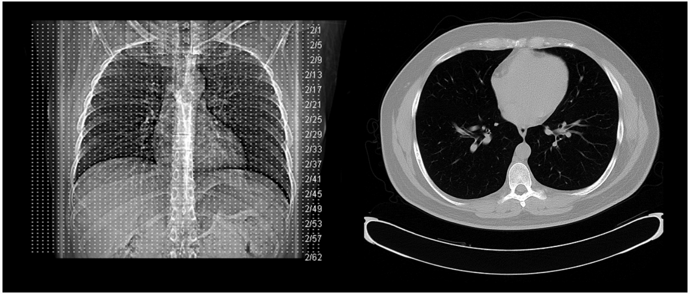
What do you mean computed tomography has something in common with deep inelastic scattering in the dipole picture??

- It's *not* the physics (X-ray attenuation in the target)
- It's *not* the experiment (CT scanner, detector rotating around the target)
- It's *not* the mathematical relationship (Radon transform)

We need to look at CT to know how to throw out all of the above, and carefully replace them with the dipole picture of deep inelastic scattering.

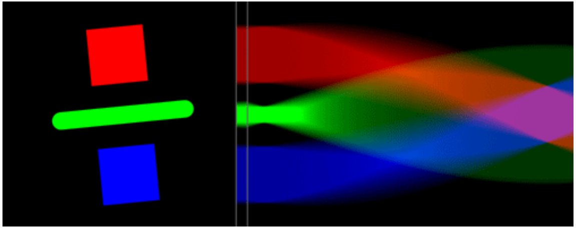
An instructive example: computed tomography (CT scan)





Tomographic reconstruction in medical imaging. However, we are interested in the mathematical theory which enables this novel shift in perspective into the structure. *The measurement never sees the target from the axial direction, yet exactly that is uncovered mathematically from the data*.

The measured data in CT is called the *sinogram*



- X-ray source from the left exposes the target which is captured by the sensor on the right.
- The source–sensor pair is rotated around the target to produce the *sinogram* in the right panel.

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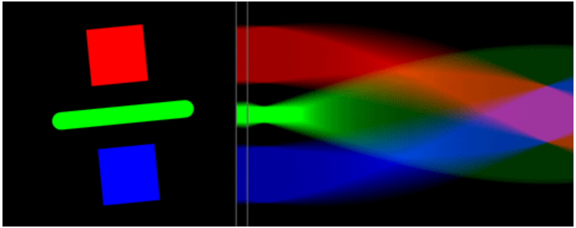


- X-ray source from the left exposes the target which is captured by the sensor on the right.
- The source–sensor pair is rotated around the target to produce the *sinogram* in the right panel.
- The mathematical inverse problem in CT is the recovery of the target structure from the measured sinogram.
    - The fundamental inverse problem of CT has nothing to do with X-rays or biology, but rather with geometry and properties of the associated integral.

Sinogram is the *Radon transform* of the target

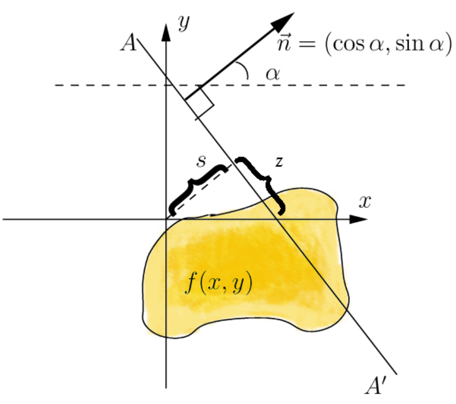
The *Radon transform* Rf is a function defined on the space of straight lines $L \subset \mathbb{R}^2$ by the line integral along each line as:

$$Rf(L) = \int_L f(x)|\mathrm{d}\mathbf{x}|.$$

Parametrizing the lines with path length z:

$$(x(z), y(z)) = \big((z \sin\alpha + s\cos\alpha), (-z\cos\alpha + s\sin\alpha)\big)$$

one has $Rf(\alpha, s) = \int_{-\infty}^{\infty} f(x(z), y(z))\mathrm{d}z.$



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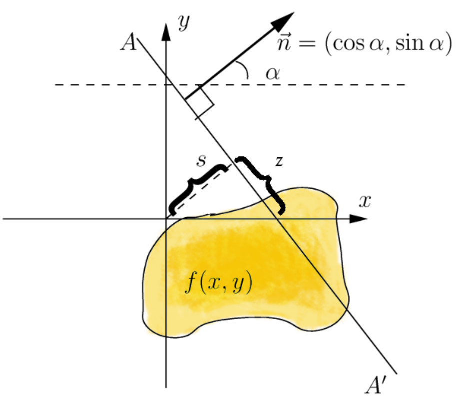
It is analytically invertible:

$$f(\mathbf{x}) = \int_0^{\pi} (Rf(\cdot,\theta) * h)\langle \mathbf{x}, \mathbf{n}_\theta\rangle)\mathrm{d}\theta,$$

with the convolution kernel h such that $\hat{h}(k) = |k|$.



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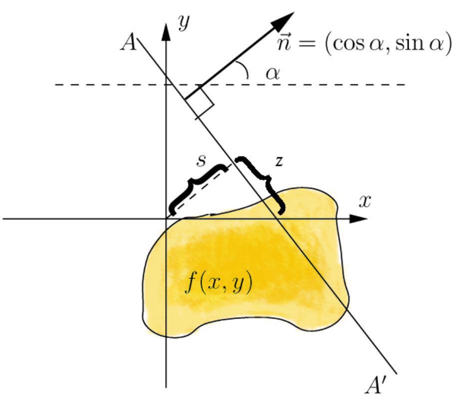
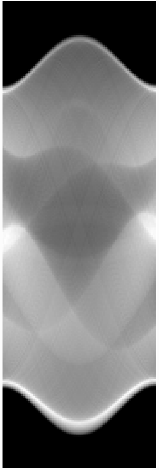


Radon transform 1917, CT patent 1963, first commercial CT scanner 1972.

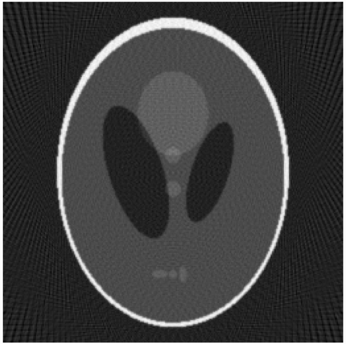
How do we apply this approach to our benefit in QCD?

Reconstruction of the Shepp–Logan phantom

Shepp–Logan phatom: a standard test image.



Radon transform of the phantom.



Inverse Radon transform (reconstruction) from the sinogram.

Back to high-energy physics

Well how do we do anything remotely similar with the dipole picture of DIS?

Reduced cross section data to constrain the dipole amplitude

Reduced cross section σ_r is defined in the dipole picture at leading order accuracy as:

$$\sigma_r(y, x, Q^2) = F_T(x, Q^2) + \frac{2(1-y)}{1+(1-y)^2} F_L(x, Q^2)$$

$$F_{L,T}(x, Q^2) = \frac{Q^2}{4\pi\alpha_{\mathrm{em}}} \sigma_{L,T}(x, Q^2)$$

$$y = \frac{Q^2}{xs}$$

$$\sigma_{T,L}(x, Q) = \frac{\sigma_0}{2} \frac{1}{4\pi} \sum_f \int_{\mathbb{R}^2} \int_0^1 \left| \psi_{T,L}^{\gamma^* \to q_f \bar{q}_f}(\mathbf{r}, Q^2, z, f) \right|^2 N(\mathbf{r}, x) \mathrm{d}^2\mathbf{r} \mathrm{d}z,$$

which is an implicit inverse problem for $N(r, x)$, conventionally solved by fitting a theoretically motivated model parametrization for N, such as the McLerran–Venugopalan model.

Mathematical observations

We want to achieve **injectivity in a functional sense** for the mapping $N(r,x) \mapsto \sigma_r(Q,x)$:

$$\sigma_r(y, x, Q^2) = F_T(x, Q^2) + \frac{2(1-y)}{1+(1-y)^2} F_L(x, Q^2)$$

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First simplifying assumptions:

- Keep x fixed (constant) in calculations. Reconstruct $N(x,r)$ at fixed x.
- Take s as constant. With these constraints $y \equiv y(Q^2)$.

Rewriting into an explicit inference problem

Define z-integrated help functions:

$$Z_{T,L}(r, Q^2) = \sum_f \int_0^1 \left| \psi_{T,L}^{\gamma^* \to q_f \bar{q}_f}(r, Q^2, z, f) \right|^2 \mathrm{d}z$$

and the reduced cross section kernel function

$$Z(r, Q^2, y) := \frac{rQ^2}{4\pi\alpha_{\mathrm{em}}} \left[Z_T(r, Q^2) + \frac{2(1-y)}{1+(1-y)^2} Z_L(r, Q^2) \right].$$

With these we can write the reduced cross section in the dipole picture as:

$$\begin{aligned}
\sigma_r(x, Q^2) &= \frac{\sigma_0}{2} \frac{Q^2}{4\pi\alpha_{\mathrm{em}}} \int_0^\infty \left[Z_T(r, Q^2) + \frac{2(1-y)}{1+(1-y)^2} Z_L(r, Q^2) \right] N(r, x) r \mathrm{d}r \\
&= \int_0^\infty Z(r, Q^2, y(Q^2)) \frac{\sigma_0}{2} N(r, x) \mathrm{d}r,
\end{aligned}$$

which now is an **integral transform** of the dipole amplitude. (Or looks like one anyway, let's try inverting it with brute force to see if it makes sense to try to do proper analysis of invertibility.)

Sidebar: Related integral transforms

The K-transform defined by

$$g(y,\nu) = \mathcal{K}_\nu[f(x);y] = \int_0^\infty (xy)^{\frac{1}{2}} K_\nu(xy) f(x) \mathrm{d}x$$

is analytically invertible. Identifying the measurement with $g(y) \sim \sigma_r(Q)$ and the dipole amplitude with $f(x) \sim N(r)$ bears a rough resemblance to the reduced cross section integral transform of the dipole amplitude.

Guess: the fact that our approach works could be related to the existence of a proper integral transform? Closer mathematical inspection needed, work in progress.

Discretizing the problem for numerical inversion

We want to numerically compute the inverse of the ”dipole cross section transform“ without having an explicit formula for the inversion. Inverse problems methods enable us to do this.

[2] The symbol ς, `\varsigma`, is the *final form* lower case sigma used at the end of a word: ”f-sigma“.

Discretizing the problem for numerical inversion

We want to numerically compute the inverse of the ”dipole cross section transform“ without having an explicit formula for the inversion. Inverse problems methods enable us to do this. Taking r on a grid $\{r_0, r_1, \dots, r_M\}$ and writing the r-integral as a Riemann sum:

$$\sigma_r^{\mathrm{d}}(x, Q^2) = \sum_{i=0}^{M} (r_{i+1} - r_i) Z(r_i, Q^2, y(Q^2)) \frac{\sigma_0}{2} N(r_i, x),$$

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we write the problem as a proper linear equation:

$$\sigma_{r,j}^{\mathrm{d}} := \sigma_r^{\mathrm{d}}(Q_j^2, x) = \varsigma_j^i \mathbf{n}_i,$$

where we absorbed the interval length $(r_{i+1} - r_i)$ into the definition of ς_j^i (the forward operator[2]), and $\frac{\sigma_0}{2}$ into $\mathbf{n}_i$ (discretized dipole amplitude). This is a discrete linear inverse problem for $\mathbf{n}_i$.

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Discrete linear inverse problem

The discrete linear inverse problem

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is an explicit reconstruction problem of the form $\mathbf{b} = \mathbf{A}\mathbf{x}$, to which we may apply standard methods, such as Tikhonov regularization.

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- The forward operator ς is now completely fixed by the (light-cone) perturbation theory and the experiment via the definition of the reduced cross section, and all non-perturbative degrees of freedom are in $\mathbf{n}_i := \frac{\sigma_0}{2} N_i$, to be solved by reconstruction.

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- With a given discretization over r and Q, and fixed quark flavors and masses, the forward operator is just a fixed matrix: $\varsigma^i_j = (r_{i+1} - r_i) Z(r_i, Q^2_j, y(Q^2_j))$, which we can precompute and store.

Solving the inverse problem numerically

Generally the discrete problem $\sigma = \varsigma \mathbf{n}$ has

- $\mathbf{n} \in \mathbb{R}^M$: M is constrained by numerical accuracy of the computation (r grid size)
- $\sigma \in \mathbb{R}^m$: m is constrained by the available data
- $\varsigma \in \mathbb{R}^{m \times M}$: matrix valued forward operator.

If $m \equiv M$ we would have a full-rank problem (and ς could have a true inverse), but in practice we have a underdetermined problem with $m < M$.

- (i.e. there is fewer data points of σ_r available than we need to have discretization points for $N(r,x)$ to have accurate calculation of the cross sections.)

Regularization methods allow us to solve underdetermined problems, and to work with data that has noise / uncertainty.

- Computing the inverse integral transform from perfect data is easier than from data that has noise. Applications like medical imaging routinely work with noisy real world data.

Numerical reconstruction and regularization

We use 1st order derivative operator Tikhonov regularization, where we minimize the cost function

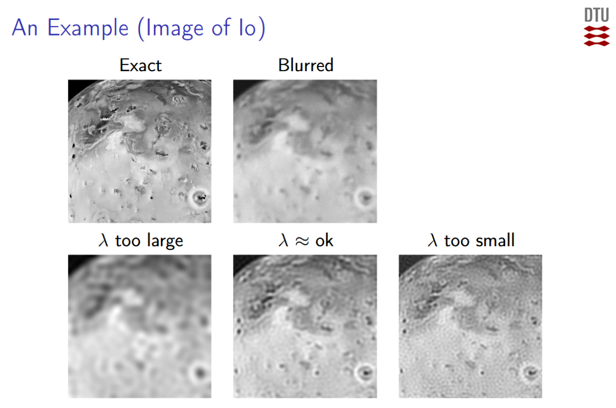
$$\underset{\mathbf{n}\in\mathbb{R}^M}{\arg\min}\,\{||\varsigma\mathbf{n}-\sigma_r||_2^2+\lambda||\frac{\mathrm{d}}{\mathrm{d}r}\mathbf{n}||_2^2\},$$

where the first is the ”data fitting term“ and the second is the regularization term. The regularization parameter λ tunes the size of the penalization of a large value for the first derivative of $\mathbf{n}$.

- Standard Tikhonov regularization would just optimize for $||\mathbf{n}||_2^2$
- 2nd order preconditioned would optimize $||\frac{\mathrm{d}^2}{\mathrm{d}r^2}\mathbf{n}||_2^2$, i.e. limit large fluctuations in the 2nd order derivative of the dipole amplitude.

Implementations of regularization algorithms by AIR Tools II and regtools:
`https://github.com/jakobsj/AIRToolsII`,
`https://www2.compute.dtu.dk/~pcha/Regutools/`

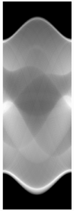
Heuristic idea of the role of the regularization parameter λ

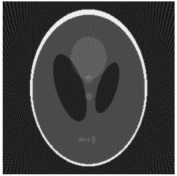


More on the choice of regularization parameter at `http://www2.imm.dtu.dk/~pcha/DIP/chap5.pdf`.

- The regularization parameter λ "tunes" the sensitivity of the algorithm for noise: too large λ smoothens too much, and too small λ allows noise to pass through to the reconstructed solution.
- Choosing λ is a critical step: this closure test is used to verify that it is chosen well enough.
- With real data we don't have a "ground truth" (exact information about the dipole amplitude we wish to reconstruct), we need an unsupervised method:
  - We use $\mathrm{err}_{\mathrm{noiseless}}$ and err_{χ^2}.
  - More robust methods: discrepancy principle, L-curve criterion, GCV criterion, NCP criterion.

Choosing a ”phantom“ of the dipole amplitude





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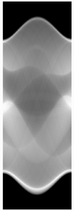

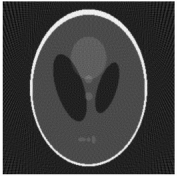

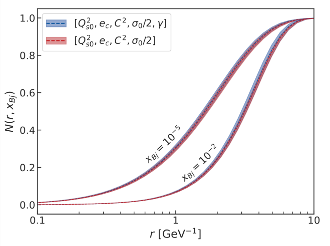

Casuga–Karhunen–
Mäntysaari reference dipole

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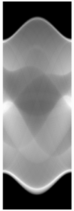
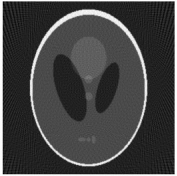
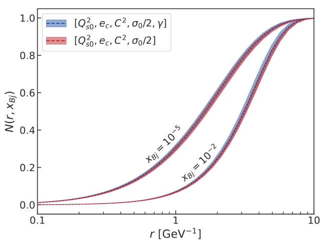
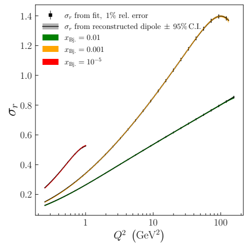




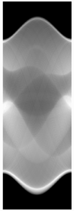


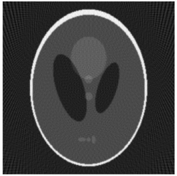
Casuga–Karhunen–Mäntysaari reference dipole

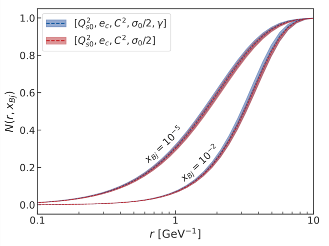


DIS reduced cross section from CKS dipole

Choosing a ”phantom“ of the dipole amplitude







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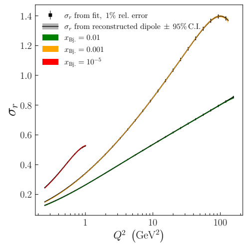

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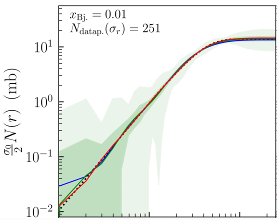


Reconstruction

Generating data for the closure test

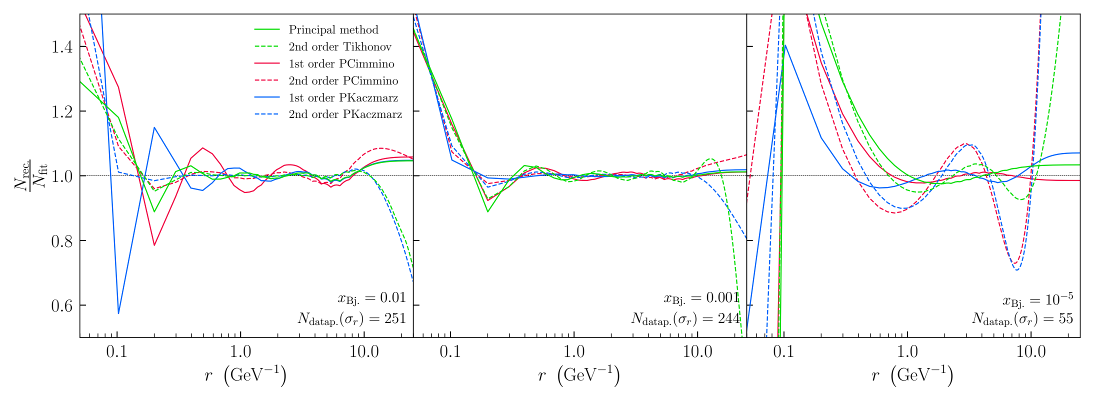
We use the four and five parameter Bayesian LO CKM inferences as the references dipoles to generate data:

- Casuga, Karhunen, Mäntysaari, Phys. Rev. **D** 109, 054018
- We generated two cross section datasets, and two corresponding forward operators:
  - Light quarks only, as in the reference CKM work.
  - Light + charm to demonstrate that the reconstruction works also with the inclusion of charm in the forward operator. Used cross section data is artificial in the sense that original fits did not include charm.

We compute in HERA kinematics with $\sqrt{s} = 318\,\mathrm{GeV}$ reduced cross section data in similar bins of Bjorken-x, but with more ample number of points in Q^2.

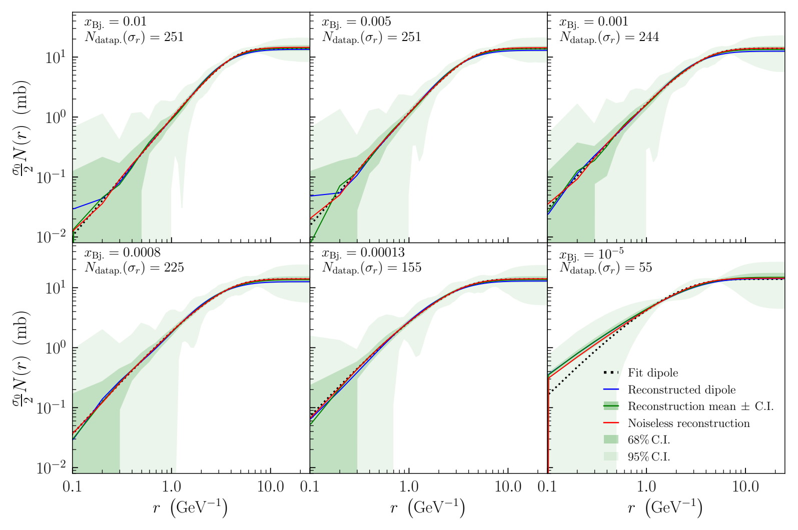
- A full rank problem with $m = M := 256$ would be the reference for an "easy" problem.
- The less data of the measurement $\sigma_r(Q^2)$ we have available, i.e. $m < M$, the more underdetermined, and harder, the problem becomes.

Comparison of regularization methods



- Comparison of six best reconstruction algorithms as shown by the ratio $\frac{N_{\mathrm{rec}}}{N_{\mathrm{fit}}}$.
- Chosen ”Principal“ method, 1st order preconditioned Tikhonov, performed the best on average.

Reconstruction results: light quarks only, 4-par-CKM-dipole



- Overall the reconstruction is working quite well.
- At small x with the fewest datapoints, the reconstruction loses accuracy.
  - $Q^2 = s x_{\mathrm{Bj}} y \leq s x_{\mathrm{Bj}}$
- Growing uncertainty at small-r reflects that the reduced cross section data is not sensitive to the dipole amplitude in that regime, which is expected.

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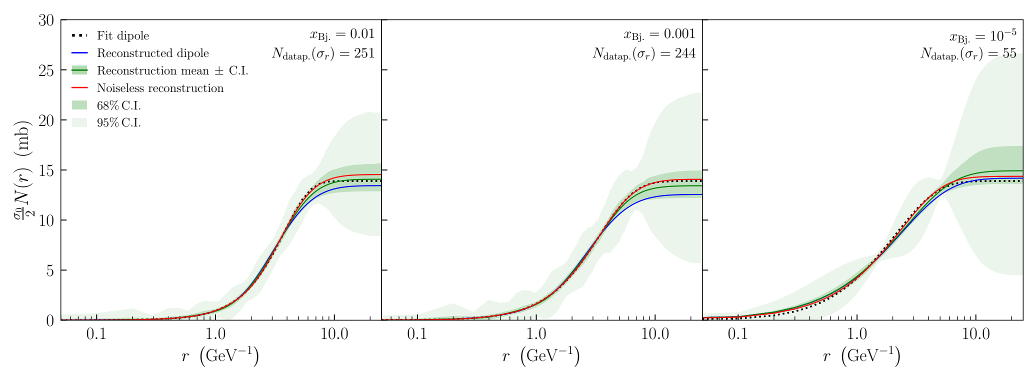
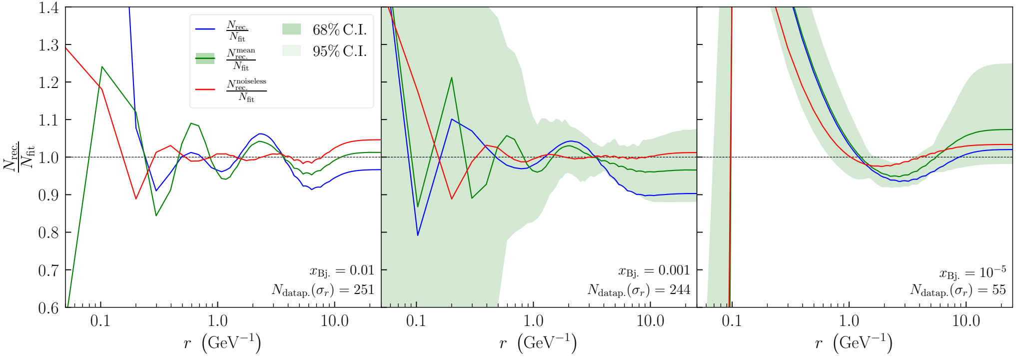


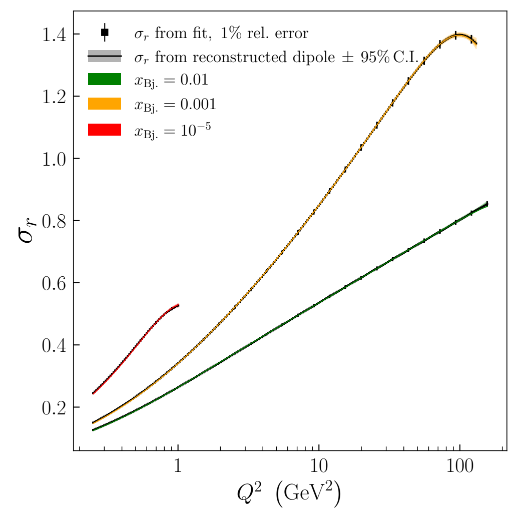
- Linear y-axis: the reconstruction is working very well in the intermediate-r regime, and captures the decay to zero at small-r at least in absolute terms.
- At large-r we see that accuracy is reduced due to the exponentially vanishing forward operator ς. With sufficient data it is working to some degree.

Relative accuracy of the reconstrution



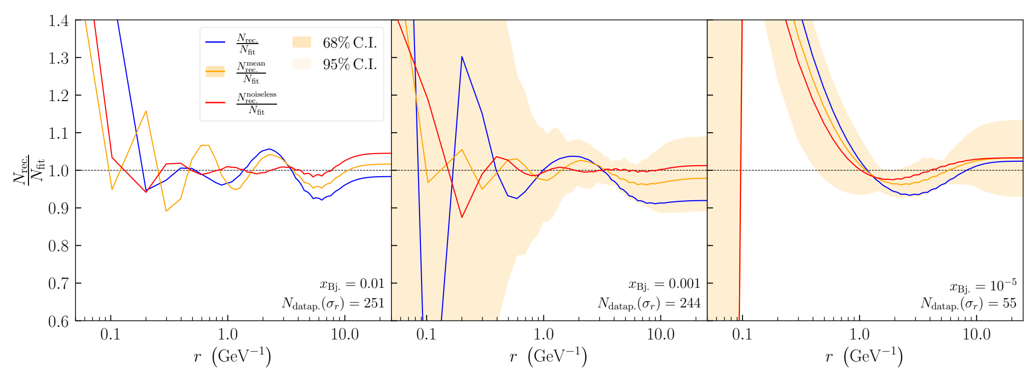
- Proof-of-principle: at least with enough data (of sufficient quantity and fidelity) this reconstructive approach is capable of reaching high accuracy, especially in the intermediate-r regime.

Cross sections from reference dipole and reconstruction



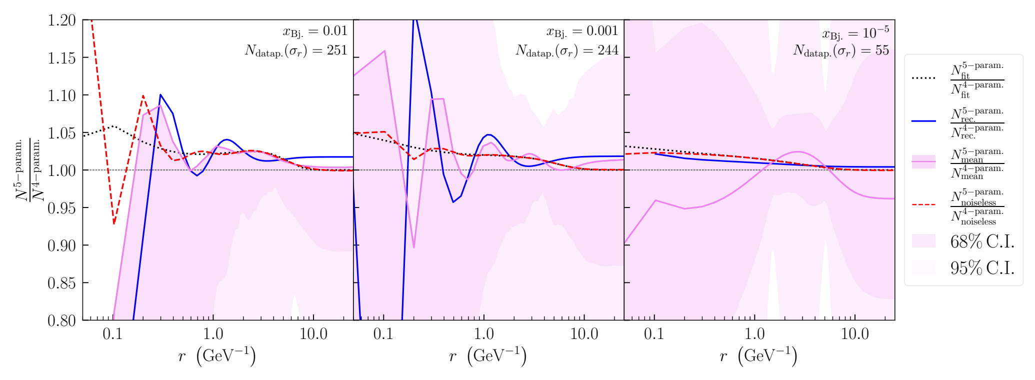
- **All** of the reconstructions from the 95% confidence intervals map into the colored 95% C.I. bands shown here [sic].
- Shows how the reduced cross section data does not constrain the dipole amplitude at small r, and only to a limited degree at large r.

Reconstruction of the 5-parameter reference dipole



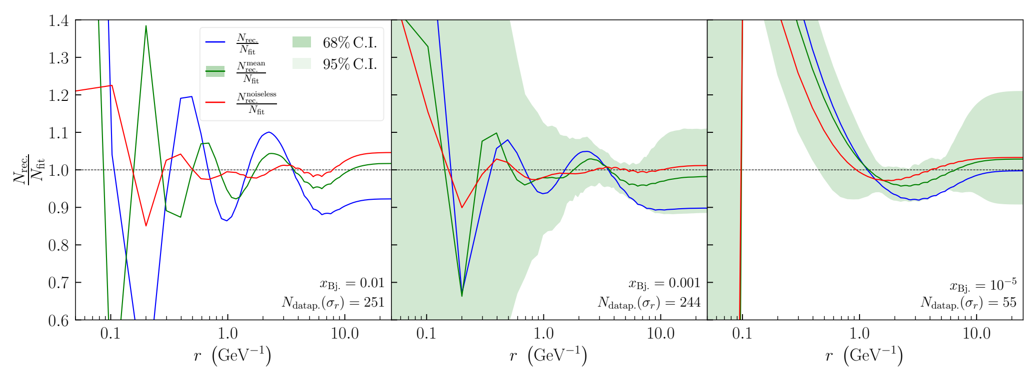
- Performance of the reconstruction is very similar for the second reference dipole as well. The algorithm is completely agnostic of the original parametrization of the reference dipole.

Can it differentiate between the two parametrizations?



- Proof-of-principle: in ”perfect“ (noiseless) conditions, the ratio is recovered quite faithfully.
- Given sufficient data the reconstruction is able to reproduce the ratio of the two reference fit dipole amplitudes. With reduced number of datapoints and the addition of noise (error) in the data, the reconstruction begins to deteriorate or fail.

Accuracy with light + charm quarks



- Addition of the charm contribution to the forward operator is a minor change that does not affect the performance of the reconstruction. The underlying problem stays linear.

Where is the Balitsky–Kovchegov equation in all of this?

Why not use the BK equation?

- We first wanted to establish a proof-of-principle of this approach without introducing more complexity:
  - Including the BK will connect the reconstructions of adjacent $x_{\mathrm{Bj.}}$-bins, making the problem two dimensional.
  - However, compared to typical image reconstruction, where the ”horizontal and vertical“ directions in the image are not fundamentally any different, here that is the case for the r and $x_{\mathrm{Bj.}}$ directions. $\implies$ little bit of a head scratcher to implement.

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  - However, compared to typical image reconstruction, where the ”horizontal and vertical“ directions in the image are not fundamentally any different, here that is the case for the r and $x_{\mathrm{Bj.}}$ directions. $\implies$ little bit of a head scratcher to implement.
- On the other hand, *if it can work* without the BK equation, we would be able to reconstruct also the evolution of the dipole without assuming a specific evolution prescription.
  - The reconstruction would provide data about the x-evolution that could be compared with the BK equation for example.

Implementing the BK equation

We are of course looking into how to do it, not least because it could help with the implementation to real data.
Roughly, the enforcement of the x-evolution could look something like:

$$\arg\min_{\mathbf{n}\in\mathbb{R}^M} \{||\varsigma\mathbf{n} - \sigma_r||_2^2 + \lambda||\frac{\partial}{\partial r}\mathbf{n}||_2^2 + \beta||\frac{\partial}{\partial x}\mathbf{n}||_2^2\},$$

where $\frac{\partial}{\partial x}\mathbf{n}$ is computed from the BK equation, and the new regularization parameter β would tune how exactly we expect the reconstruction to adhere to the prescription of the evolution as given by the BK equation. (which isn't exact due to step size / data availability in x, finite order in α_s, mean field approximation, running coupling, resummation assumptions,...).

HERA data

What about HERA data!?

Available HERA data

We need $\sigma_r(Q^2, x, y = y(s, x, Q^2))$ in bins of fixed s and x.

HERA data[a] is available in four bins of $\sqrt{s}$:

$\sqrt{s}$	N
318.1	644
300.3	112
251.5	260
224.9	210

Data at lower $\sqrt{s}$ almost exclusively at large x, and bins are small.

[a] H1 and ZEUS Collab., H. Abramovicz et al., Eur.Phys.J.C75 (2015) 12, 580

At $\sqrt{s} = 318\,\mathrm{GeV}$, $x_{\mathrm{Bj.}} <= 0.08$, all bins with ≥ 6 points in Q^2:

$x_{\mathrm{Bj.}}$	$N(\sigma_r(\sqrt{s}, x_{\mathrm{Bj.}}))$
0.00013	8
0.0002	9
0.00032	11
0.0005	15
0.0008	19
0.0013	18
0.002	21
0.0032	24
0.005	24
0.008	23
0.013	30
0.02	35
0.032	33
0.05	30
0.08	30

Towards global analysis as a multimodal inverse problem

How do we use all possible data simultaneously? And what about the charm production data?
We can stack the reconstruction problems and the algorithm doesn't care:

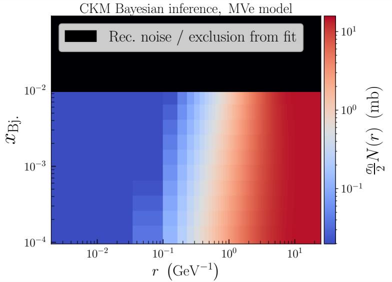
$$\begin{bmatrix} \sigma_r^{\text{incl.}}(x, Q^2, s_1) \\ \sigma_r^{\text{incl.}}(x, Q^2, s_2) \\ \sigma_r^{\text{incl.}}(x, Q^2, s_3) \\ \sigma_r^{c\bar{c}}(x, Q^2, s_{c\bar{c}}) \end{bmatrix} = \begin{bmatrix} \varsigma_r^{\text{incl.}}(Q^2, s_1) \\ \varsigma_r^{\text{incl.}}(Q^2, s_2) \\ \varsigma_r^{\text{incl.}}(Q^2, s_3) \\ \varsigma_r^{c\bar{c}}(Q^2, s_{c\bar{c}}) \end{bmatrix} \mathbf{n}(r, x),$$

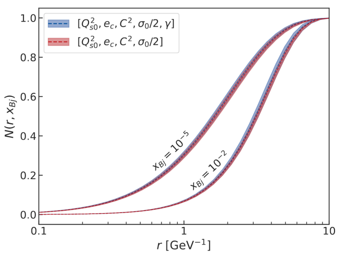
i.e. we can compute a new forward operator for each $s \in \mathbb{R}$, and include the corresponding data in the reconstruction. This also enables the inclusion of the charm production data: the forward operator is computed only with $f \equiv c$. *Work in progress.*

HERA data preliminary results disclaimer

- Testing the implementation of the method paper without new improvements like enforcing non-negativity.
- **Very** limited quantity of HERA data in the form we need: only using the largest $\sqrt{s}$ set.
  - Some of the data is at high $Q^2 \geq 400\,\mathrm{GeV}^2$, but we use them to have any hope of having enough data points.
- Not including charm production data (yet!).
- Not leveraging the BK equation which could help with the limited data situation.
- Forward operator is only LO: we don't know how this affects the reconstruction ("model uncertainty" in inverse problems): heuristically reconstructed $N_{\mathrm{rec}} \sim \varsigma_{\mathrm{LO}}^{-1}(\sigma_r^{\mathrm{HERA}})$, so the reconstruction compensates for everything *not exactly* the LO dipole picture: beyond LO in α_s, next-to-eikonal effects, large Q^2 effects(?), and probably much more.
- Rudimentary implementation of the unsupervised regularization, we should try to get the better methods working. We might not be finding the optimal λ, which could cause surplus noise in the reconstruction, or over-smoothing.

Visualizing the dipole amplitude data: Reference dipole





- The CKM reference dipole shown in a $(r, x_{\mathrm{Bj.}})$ plane.
- BK equation describes the evolution in $x_{\mathrm{Bj.}}$.
- Initial condition of the CKM dipole amplitude parametrized according to the MVe model:

$$N(r, x = 0.01) = 1 - e^{-\frac{(r^2 Q^2_{s,0})^{\gamma}}{4} \ln\left(\frac{1}{r\Lambda_{\mathrm{QCD}}} + e_c e\right)}.$$

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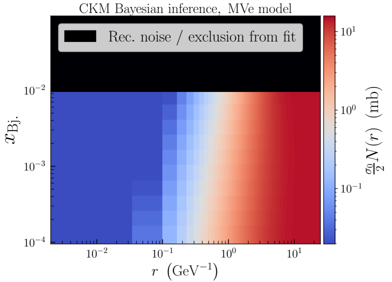

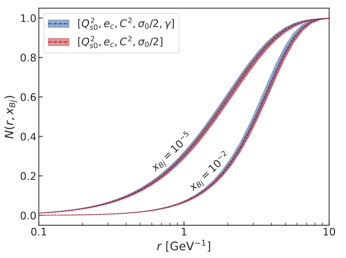

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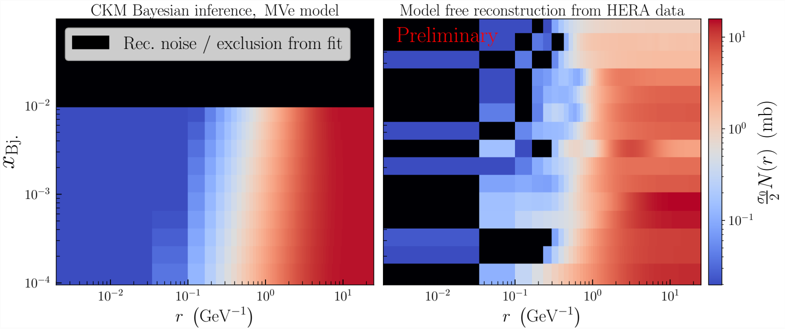
- Uniform target size $\frac{\sigma_0}{2}$ has been assumed to be reasonable for inclusive DIS cross section data: transverse momentum exchange is not measured, so it is thought that inclusive DIS is not sensitive to the target size.
  - Note normalization: upper figure shows $\frac{\sigma_0}{2}N(r, x)$ in log scale, lower has linear y-axis with dipole normalized to unity.

Preliminary HERA data reconstruction, only $\sqrt{s} = 318\,\mathrm{GeV}$



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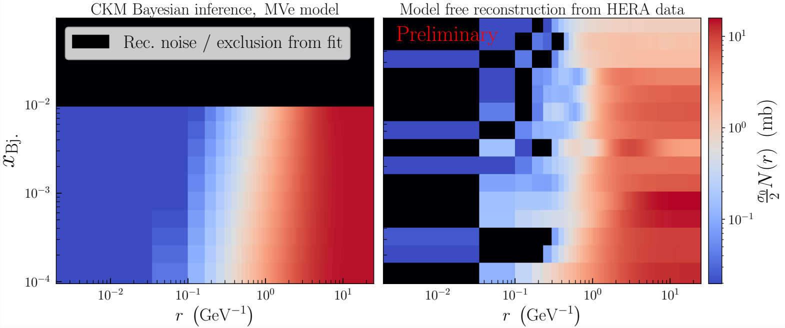

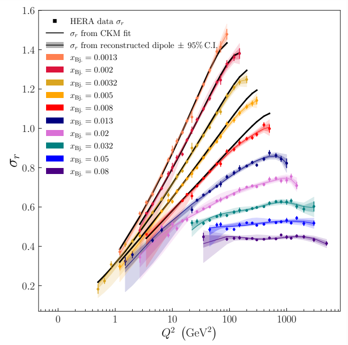

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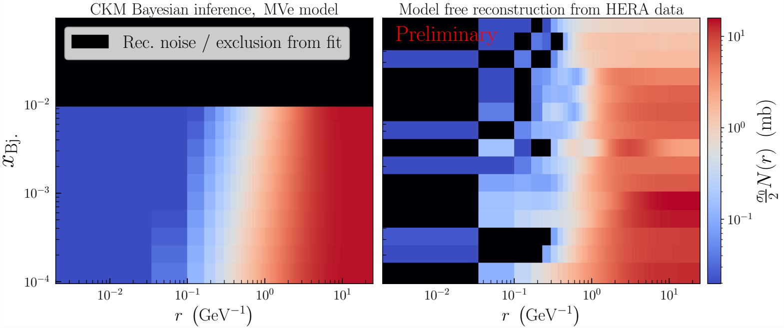

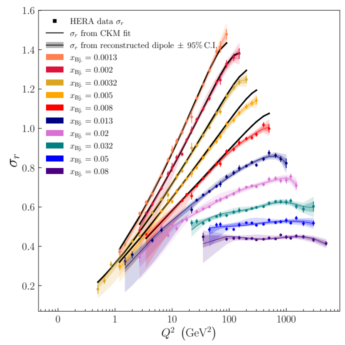

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  - Once we establish accuracy of the reconstruction, we can extract $Q_s(x)$ from HERA data without an assumed model parametrization(!) (Perhaps unintegrated gluon distribution, $\sigma_0(x_{\mathrm{Bj.}})/2$?)

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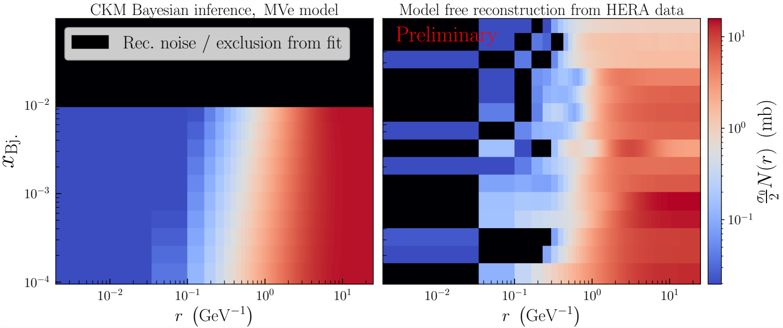

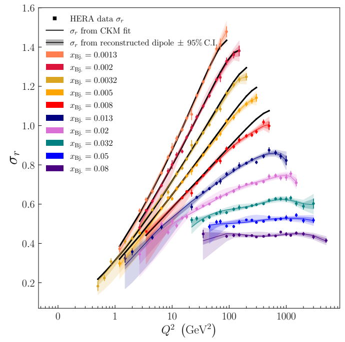

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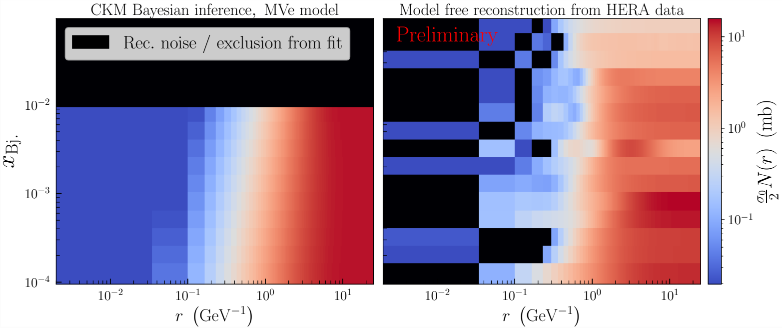

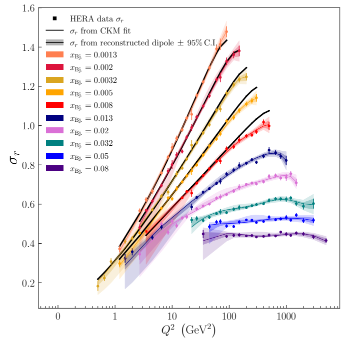

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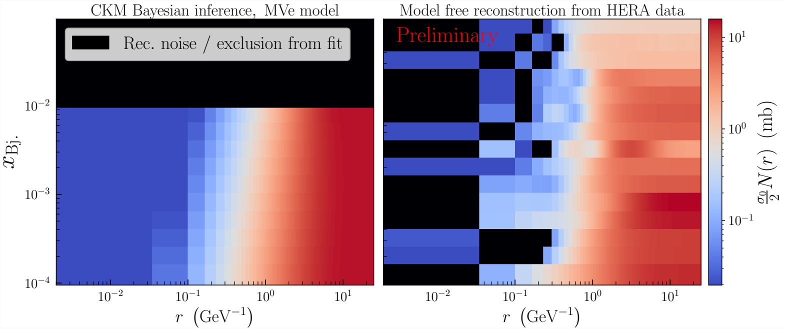

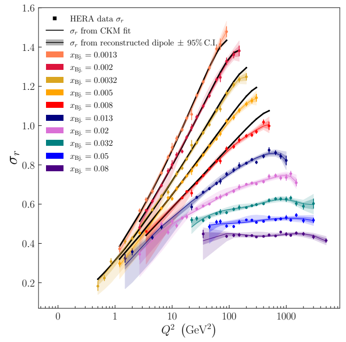

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- Vision: Inverse problems inference can hopefully enable the discovery of new features from data! (Physics that hasn't been included in previous models: non-perturbative effects etc.)
- If we were inverting a known integral transform like the Fourier or Radon transform, would this be surprising as an approach?
  - Homework for us to mathematically establish properties for this transform.

Conclusions

- We wrote the dipole picture of DIS into an integral transform of the dipole amplitude, and showed that it can be inverted with standard numerical inverse problems methods.
  - *Even without having any formula for the inverse!*

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  - Ultimate goal to enable generalized indirect measurement in high-energy physics applications.

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  - Analysis of the inverse problem can even inform what would be an ”efficient“ measurement.

$\therefore$ If there is *data* and a *theory that can fit that data* $\implies$ basis for an inverse problem.
(To develop a more general approach of inference than fitting model parameters.)

Thank you for your attention!

Questions?

Contact:
henri.j.hanninen@jyu.fi

Slides:
https://hhannine.github.io/